Matching with Externalities

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Abstract

We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to allow for externalities, we establish that stable matchings exist when agent choices satisfy substitutability. In addition, we show that the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, remain valid despite the presence of externalities even though the standard fixed-point techniques do not apply. Furthermore, we establish novel comparative statics on externalities.

1 Introduction

Externalities are present in many two-sided markets. For instance, couples in a labor market pool their resources as do partners in legal or consulting partnerships. As a result, the preferences of an agent may depend on the contracts signed by the partner(s). Likewise, a firm's hiring decisions are affected by how candidates compare to competitors' employees. Finally, because of technological requirements of interoperability, an agent's purchase decisions may change because of other agents' decisions.

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In this paper, we incorporate externalities into the stable matching theory of Gale and Shapley (1962) and Hatfield and Milgrom (2005).[1] We refer to the two sides of the market as buyers and sellers. Each buyer-seller pair can sign many bilateral contracts. Furthermore, each agent is endowed with a choice function that selects a subset of contracts from any given set conditional on other agents' contracts. We build a theory of matching with externalities that both extends to this more general setting some of the key insights of the classical theory without externalities, such as the existence of stable matchings and Gale and Shapley's deferred acceptance (or cumulative offer) algorithm and establishes new insights, including comparative statics on externalities.

Our theory is built on a substitutes condition that extends the classical substitutes condition to the setting with externalities.[2] We require that each agent rejects more contracts from a larger set (as in the classical substitutes condition) and also that each agent rejects more contracts conditional on a matching that reflects better market conditions for his side of the market. We formalize the latter idea in two steps. A matching reflects better market conditions for one side of the market than another matching whenever the first matching is chosen by agents on this side of the market from a larger set conditional on a matching while the second matching is chosen by the agents from a smaller set conditional on the same matching. The second matching then reflects worse market conditions. Furthermore, we also say that a matching reflects better market conditions for one side of the market than another matching whenever the first matching is chosen by agents on this side of the market from some set conditional on some matching while the second matching is chosen by these agents from a smaller set conditional on a matching that reflects worse market conditions. When there are no externalities, this substitutes condition reduces to the classical gross substitutes condition of Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

We start by proposing an algorithm akin to the deferred acceptance algorithm for the setting with externalities, which may be important in potential market design applications. In particular, the algorithm can also be viewed as a new auction that performs well in the pres-

[1] Let us stress that even though we derive our results in a general many-to-many matching setting with contracts, the results are new in all special instances of our setting, including many-to-one and one-to-one matching problems.

[2] We formulate most of our results in terms of choice functions satisfying the *irrelevance of rejected contracts*. A choice function satisfies the irrelevance of rejected contracts if removing a rejected contract does not change the chosen set conditional on the same matching. When there are no externalities, this condition reduces to the one used in Aygün and Sönmez (2013). This is a basic rationality axiom: it is satisfied tautologically whenever agents' choice can be rationalized through a strict preference ordering.

ence of externalities.[3] Since an agent's choice depends on others' matching, we keep track not only of which contracts are available but also of the reference matchings that agents on each side use to condition their choice. The construction requires care because after the reference matching has changed an agent may want to go back to a contract that is already rejected. To ensure that this does not happen, we construct the initial reference matchings in a preliminary phase of the algorithm.[4] Relatedly, we cannot stop the algorithm as soon as the set of available contracts converge: we need to continue until the reference matchings converge as well. Our construction of initial reference matchings ensures that subsequent reference matchings change in a monotonic way with respect to the "better market conditions" preorder, thus ensuring that from some point on the reference matchings belong to the same equivalence class. While these equivalence classes might consist of many matchings, we further show that the algorithm converges to one of them and never cycles among the members of the same equivalence class.

Our first main result shows that our algorithm always converges to a stable matching when choice functions satisfy substitutability (Theorem 1), and hence that stable matchings exist. We focus on the classical short-sighted stability concept in which each agent assumes that other agents do not react to his or her choice. Our results, however, are applicable to many other stability concepts including far-sighted ones because we formulate the results in terms of agents' choice behavior and not in terms of their preferences. As we discuss in Section 3.1, agents' choice behavior captures both agents' preferences and their conjectures about the reactions of other agents' to choices. The resultant synthesis of short-sighted and far-sighted stability is one of the main conceptual contributions of our paper.[5]

Our second main result is a comparative statics on the strength of externalities and substitutes. Comparing two profiles of choice functions, we say that *substitutes are stronger* when agents reject more. In addition, we say that a reference choice function has *weaker externalities* than another choice function when the reference choice function reflects better market conditions (when the market conditions are measured by the reference choice function) than the

[3] See, e.g., Abdulkadiroğlu and Sönmez (2003) and Sönmez and Switzer (2013) for market design applications of deferred acceptance, and Kelso and Crawford (1982) and Hatfield and Milgrom (2005) for the relationship between deferred acceptance and ascending auctions.

[4] The cumulative offer phase of the algorithm builds on the approach of Fleiner (2003) and Hatfield and Milgrom (2005). The preliminary phase of the algorithm has no forerunners. It may be omitted if there is an underlying lattice structure on the set of all matchings; in general, however, such a lattice structure does not exist and neither do side-optimal matchings.

[5] While the study of stability in terms of choice behavior is well established (see e.g. Aygün and Sönmez, 2013), we believe that this conceptual point is new. The choice-based approach allows us to also consider agents whose choice behavior cannot be represented in terms of preferences as long as this choice behavior satisfies the rationality postulate discussed in footnote 2.

other choice function. This comparison of the strength of externalities satisfies some natural properties: for instance, the choice function exhibiting no externalities has weaker externalities than any other choice function. We prove that agents on one side of the market face better market conditions as their side of the market exhibits stronger substitutes and weaker externalities and they face worse market conditions if the other side of the market exhibits stronger substitutes and weaker externalities (Theorem 5).

In addition to these results, we extend the classical theory of matching to the setting with externalities. In Section 5, we first show that every stable matching is Pareto efficient (Theorem 3). Then we analyze the existence of side-optimal stable matchings, that is, matchings that represent the optimal market conditions. A side-optimal stable matching exists under the additional assumption that there exists a side-optimal matching (Theorem 4). This additional assumption is satisfied trivially in finite settings without externalities, where the existence of side-optimal stable matchings was established already by Gale and Shapley (1962).

Furthermore, we study vacancy-chain dynamics. What are the welfare implications of an agent leaving the market? We show that when agents recontract according to an algorithm akin to the deferred acceptance algorithm, all agents on the same side face better market conditions and all agents on the other side face worse market conditions (Theorem 6). In the setting without externalities and when agents on one side of the market can sign only one contract, the corresponding results have been proven by Kelso and Crawford (1982) and Crawford (1991). Similarly, our results generalize those of Blum, Roth, and Rothblum (1997) and Hatfield and Milgrom (2005), none of whom looked at the setting with externalities.

We also generalize the rural hospitals theorem of Roth (1986), which states that each hospital gets the same number of doctors in each stable matching in many-to-one matching without externalities (in Appendix A). Our generalization allows different contracts to have different weights that may depend on the quantity, price, or quality of the contracts. For this purpose, we introduce a general law of aggregate demand. An agent's choice function satisfies *the law of aggregate demand* if the weight of contracts chosen from a set conditional on a reference matching is greater than the weight of contracts chosen from a subset conditional on a matching that has worse market conditions than the reference matching. We show that when choice functions satisfy the law of aggregate demand in addition to the aforementioned properties, all stable matchings have the same weight for every agent (Theorem 7). When there are no externalities, this law of aggregate demand reduces to the monotonicity condition of Fleiner (2003).

To the best of our knowledge, our development of comparative statics and results such as the rural hospitals theorem with externalities have no forerunners in the literature analyzing externalities in the setting of Gale and Shapley (1962). We thus contribute to the matching literature by showing how one can incorporate externalities into standard models of matching, including matching with contracts,[6] by offering new insights, and by showing that many of the insights of the classical literature remain valid in the presence of externalities.[7]

On the other hand, our existence result contributes to a rich literature analyzing the existence and nonexistence results in matching with externalities. In an early influential paper, Sasaki and Toda (1996) showed that stable one-to-one matchings need not exist. Their insight led the subsequent literature to take one of two routes: to modify the stability concept, or to impose assumptions on agents' preferences. Sasaki and Toda's seminal paper belongs to the first strand of literature. They focused on a weak stability concept that allows a pair of agents to block a matching only if they benefit from the block under all possible rematches of the remaining agents. They show that such weak stable matchings exist.[8] In contrast, our paper uses the standard stability concept of Gale and Shapley (1962) and the literature on matching without externalities.[9] We guarantee the existence of stable matchings not by modifying the stability concept but by imposing assumptions on preferences in line with the standard approach of restricting attention to substitutable preferences. While we primarily focus on the standard (short-sighted) stability concept, our results are applicable to many other stability concepts including Sasaki and Toda's and other far-sighted concepts (see Section 3.1).

The second strand of the literature analyzes the standard stability concept.[10] Prior work in

[6]The matching with contracts approach has not only been useful as a theoretical tool but also as a practical tool to design markets. For example, see Sönmez and Switzer (2013); Sönmez (2013); Yenmez (2014). It has also been extended to the many-to-many matching and more general settings without externalities, see e.g. Ostrovsky (2008). In particular, Ostrovsky showed that stable matchings exists even in the presence of well-behaved complementarities among inputs and outputs. In a recent work, Alva and Teytelboym (2016) study supply chains in which inputs can be complements.

[7]In fact, our main comparative statics result is new even in the setting without externalities as is our synthesis of classical and far-sighted stability.

[8]The rich subsequent literature, e.g., Chowdhury (2004); Hafalir (2008); Eriksson, Jansson, and Vetander (2011); Chen (2013); Gudmundsson and Habis (2013); Salgado-Torres (2011a,b)— maintained the focus on the existence question while refining Sasaki and Toda's weak stability concept by varying the degree to which the rematches of other agents penalize the blocking pair. Bodine-Baron, Lee, Chong, Hassibi, and Wierman (2011) analyze a related weak stability concept in a setting with peer effects.

[9]In line with this literature, a set of agents forms a blocking coalition if it benefits them in the absence of any reaction from the remaining agents. Note that the question of how other agents react to the formation of a blocking coalition is important whether externalities are present or not. In particular, even in the absence of externalities, one might entertain an alternative solution concept in which an agent might be unwilling to enter a blocking coalition if she is concerned that doing so will trigger a chain of events that will lead her to losing a partner she blocks with.

[10]We follow this second approach. As discussed above, we also go beyond this second approach by offering a

this second strand of the literature identified several assumptions under which stable matchings exist. Particular attention has been devoted to externalities among couples (Dutta and Massó, 1997; Klaus and Klijn, 2005; Kojima, Pathak, and Roth, 2013; Ashlagi, Braverman, and Hassidim, 2014), to peer effects among students matched to the same college (Dutta and Massó, 1997; Echenique and Yenmez, 2007; Pycia, 2012; İnal, 2015), and to student assignment problem with neighbors (Ashlagi and Shi, 2014; Dur and Wiseman, 2015). We are not restricting our attention to either of these types of externalities.

Our contribution on the existence of stable matchings is closest to the few papers that look at standard stability in the general matching problem with externalities. Bando (2012; 2014) studies many-to-one matching allowing externalities in the choice behavior of firms (agents who match with potentially many agents on the other side) but not of workers; he further assumes that each firm's choice function depends on the matching of other firms only through the set of workers hired by other firms, and imposes several other elegant assumptions on firms' choice behavior. Under these assumptions, he proves the existence of stable matchings and analyzes the deferred acceptance algorithm. In his setting there is no need to keep track of the reference matchings in the deferred acceptance algorithm (and hence no need for the preliminary phase that constructs the initial reference matchings), and his algorithm terminates as soon as there are no rejections. In another related work, Teytelboym (2012) looks at externalities among agents in a component of a network and shows that a stable matching exists provided agents' preferences are aligned in the sense of Pycia (2012). Finally, Fisher and Hafalir (2016) consider a setting in which each agent cares only about the level of externality in the overall economy (such as pollution) and study the existence of stable matchings when there are such aggregate externalities.[11]

Our work is also related to the exploration of efficiency in markets with externalities (see, e.g., Pigou (1932); Chade and Eeckhout (2014); Watson (2014)); while this literature focuses on efficiency, we focus on stability. Furthermore, two of our examples show the applicability of our results to the analysis of dynamic matching.[12]

synthesis of standard and far-sighted approaches to stability.

[11]Also of note is Uetake and Watanabe (2012) who provide an empirical analysis of firm mergers using a matching model with externalities, and Mumcu and Saglam (2010) who analyze when all matchings in the non-empty collection of top matchings are stable. Baccara, Imrohoroglu, Wilson, and Yariv (2012) analyze stable one-sided allocations with externalities. Hatfield and Kominers (2015) study the existence of competitive equilibria in a multilateral matching setting with externalities. Leshno (2015), a work in progress, looks at large matching markets.

[12]For the analysis of dynamic matching see e.g. Ünver (2010), Pycia (2012), Kurino (2014), and Kotowski (2015).

2 Examples

In this section, we provide a few examples to motivate and illustrate our work. As it is well known, the existence of stable matchings is not guaranteed in the presence of externalities (see Example 5). However, the examples in this section satisfy the substitutes condition that guarantees the existence of stable matchings (in addition to a mild axiom). We come back to these examples after formally defining the substitutes condition.

We first present two motivating examples and then a simpler but more abstract illustrative example. Additional examples are provided in Section 7. For simplicity, we consider only one side of the market in our examples. One could model the other side in the same or a different way because we impose no assumptions relating the choice behavior of agents across sides.

Example 1. [Couples in a Local Labor Market][13] Agents on one side of the market represent workers and agents on the other side of the market represent firms. Workers are either single or are members of exogenously married couples. The labor participation decision of a married man depends on the job of his wife: the better the job she has, the more selective he becomes. In other words, the outside option of not working becomes more attractive when a man's wife has a better job. We assume that there are no externalities for firms (whose preferences satisfy the standard substitutes condition) or the single workers.

This example can be generalized such that there are externalities for both married men and women. Furthermore, any two agents can be married, so we do not need a two-sided structure for the workers.

An important extension of this example encompasses labor markets in which members of a group—e.g. a minority group or gender group—are more likely to participate in a profession if other members of this minority participate in it as well.

Our theory also applies to situations in which market participants care about the relative standings of their partners.

Example 2. [Relative Rankings in Hiring] Agents on one side of the market represent colleges and agents on the other side represent academics in a particular field. For each college i and each academic j the productivity of j at i is denoted by $\lambda(i, j) \geq 0$. For simplicity, assume that no two academics have the same productivity at a college. Now, suppose that each college hires at most two academics in the field considered, and that it wants to hire at least one academic and would like to hire another one only if his or her productivity is at least as

[13]We are grateful to Michael Ostrovsky for suggesting a simple version of this example.

high as the productivity of all academics in at least half of the other colleges. Formally, the choice function $c_i(X_i|\mu)$ of college i is as follows: from choice set X_i, the college chooses the academic $j \in X_i$ with highest productivity $\lambda(i, j)$, and it chooses a second academic $j' \in X_i$ if and only if $\lambda(i, j')$ is greater than or equal to the productivity of all academics in at least half of the other colleges under matching μ. More generally, we can fix $k \in [0, 1]$ and assume that college i chooses a second academic $j' \in X_i$ if and only if $\lambda(i, j')$ is greater or equal than the productivity of academics in at least a fraction k of other colleges.[14]

Example 3. Suppose that there are two sellers s_1 and s_2 and two buyers b_1 and b_2. Seller s_1 and buyer b_1 can sign contract x_1 and seller s_1 and buyer b_2 can sign contract x_2. Seller s_2 can sign contract x_3 with buyer b_2 only.

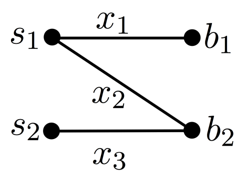

Figure 1: Contractual structure in Example 3.

Buyer b_1 wants to sign contract x_1 regardless of the contracts signed by buyer b_2. Buyer b_2 signs contract x_2 whenever it is available but signs contract x_3 only when contract x_2 is not available depending on whether buyer b_1 and seller s_1 sign contract x_1 or not. More precisely, $c_{b_2}(\{x_3\}|\emptyset) = \{x_3\}$ and $c_{b_2}(\{x_3\}|\{x_1\}) = \emptyset$. Here, the first equation means that buyer b_2 chooses contract x_3 when it is the only available contract conditional on contract x_1 not being signed and the second equation means that buyer b_2 does not choose contract x_3 if contract x_1 is signed.

Choice functions are summarized by the following tables where columns are indexed by the set of available contracts and rows are indexed by the set of contracts signed by the other buyer.

[14] We can alternatively include this fraction the college whose choice function despite the self-referentiality of doing so. While we focus our discussion on non-self-referential situations, we can in general allow the choice function of an agent to depend on this agent's choice. Note that the colleges do not need to know the productivity of all academics; for instance, in the base version of the example a college's choice function is well-defined as soon as they know the median college productivity.

	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_2,x_3\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_2\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_3\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\emptyset)$	$\{x_1\}$	$\emptyset$

	$\{x_2,x_3\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_2\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$c_{b_2}(\cdot\|\emptyset)$	$\{x_2\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$

Table 1: Buyers' choice functions in Example 3.

3 Model

There is a finite set of agents $\mathcal{I}$ partitioned into buyers, $\mathcal{B}$, and sellers, $\mathcal{S}$, $\mathcal{B} \cup \mathcal{S} = \mathcal{I}$. Agent i's type is denoted as $\theta(i) \in \{b,s\}$. With a slight abuse of notation, θ also denotes one side of the market, so $\theta \in \{b,s\}$. If θ is a type, then $-\theta$ is the other type, that is, $-b \equiv s$ and $-s \equiv b$. Agents interact with each other bilaterally through contracts. Each contract x specifies a buyer $b(x)$, a seller $s(x)$, and terms, which may include prices, salaries and fringe benefits. There exists a finite set of contracts $\mathcal{X}$. For any $X \subseteq \mathcal{X}$, X_i denotes the maximal set of contracts in X involving agent i, that is $X_i \equiv \{x \in X : i \in \{b(x), s(x)\}\}$. Similarly, X_{-i} denotes the maximal set of contracts not involving agent i, that is, $X_{-i} \equiv X \setminus X_i$. We refer to all sets of contracts as matchings. We embed any quota constraints in agents' choice behavior. For instance, we model one-to-one matching markets by assuming that each agent chooses at most one contract from any choice set. Thus, examples of our setting include standard one-to-one and many-to-one matching problems with and without transfers.[15]

Each agent i has a choice function c_i, where $c_i(X_i|\mu_{-i})$ is the set of contracts that i chooses from X_i given that μ_{-i} is the set of contracts signed by the other agents on the same side.[16] We expand the domain of the choice function so that $c_i(X|\mu) = c_i(X_i|\mu_{-i})$. Let $r_i(X|\mu) \equiv X_i \setminus c_i(X|\mu)$ be the set of contracts rejected by agent i from X_i given matching μ. Similarly define $C^\theta(X|\mu) \equiv \cup_{i\in\theta} c_i(X|\mu)$ and $R^\theta(X|\mu) \equiv \cup_{i\in\theta} r_i(X|\mu)$ to be the set of contracts chosen and rejected from set X by side θ given matching μ, respectively. Note that for any $X, \mu \subseteq \mathcal{X}$ and θ, $C^\theta(X|\mu)$ and $R^\theta(X|\mu)$ form a partition of X since every contract involves exactly one agent from each side of the market. A **matching problem** is a tuple $(\mathcal{B}, \mathcal{S}, \mathcal{X}, C^b, C^s)$.

[15] Without affecting any of the results, we could alternatively model one-to-one matching and other matching environments with quota constraints by assuming that only some sets of contracts are feasible matchings. This alternative route is straightforward if agents condition their choice behavior on any sets of contracts rather than on feasible matchings. As is usual in models of matching with contracts, in applications with transfers, we assume that there is a lowest monetary unit.

[16] We could allow choice functions c_i that depend not only on X_i and μ_{-i} but also on μ_i (that is the set of contracts signed by i) with no change in our proofs. See footnote 14 for an example of when such self-referentiality is natural.

Matching μ is **individually rational** for agent i if $c_i(\mu|\mu) = \mu_i$. Less formally, given the contracts of other agents on the same side, agent i wants to keep all of her contracts. A buyer i and seller j form a **blocking pair** for matching μ if there exists a contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}|\mu) \cap c_j(\mu \cup \{x\}|\mu)$. In words, a pair can block a matching if they can sign a new contract that both of them like. Matching μ is **stable** if it is individually rational for all agents and there are no blocking pairs. This stability concept is identical to pairwise stability studied in settings without externalities (Roth and Sotomayor, 1990). As in the standard settings without externalities, stability defined in terms of individual and pairwise blocking is equivalent to group stability when choice rules are substitutable; see Appendix B. Defining stability in terms of agents' choices rather than preferences allows us to be agnostic whether blocking agents expect no further reaction to their blocking, as in canonical stability concepts, or whether blocking agents have more complex expectations about the consequences of them blocking; see Section 3.1.

We illustrate this stability notion using Example 3. Suppose that there are no externalities for sellers and that they choose all available contracts, that is, $C^s(X|\mu) = X$ for any set of contracts X and μ. In this example, $Y = \{x_1, x_2\}$ is a stable matching. First of all, it is individually rational: buyer b_1 always wants to keep contract x_1, buyer b_2 also wants to sign contract x_2, and, likewise, seller s_1 wants to keep both contracts. Furthermore, there are no blocking pairs. The only potential blocking pair is seller s_2 and buyer b_2 with contract x_3. But buyer b_2 does not want to sign contract x_3 given contract x_1, i.e., $x_3 \notin c_{b_2}(Y \cup \{x_3\}|Y)$. Therefore, Y is a stable set.

3.1 Standard and Far-Sighted Stability

We take choice functions as primitives of our model. This approach allows us to offer a unified theory of stability that does not depend on blocking agents' hypothesis on how other agents react.[17] In general, when agents have preferences over matchings (sets of contracts) then these preferences and agents' predictions of how others will react to the changes in a matching allows us to construct the choice functions. In particular, while we focus on standard stability in which agents assume that their choice does not trigger chains of reactions by others, the general choice formulation we study implies that our results are equally applicable to theories of far-sighted stability. In the rest of the discussion, we give two simple examples of how

[17] This approach has many other benefits (Chambers and Yenmez, 2013) and it has been used in a matching context before (Fleiner, 2003; Alkan and Gale, 2003; Hatfield and Milgrom, 2005; Aygün and Sönmez, 2013).

agents' preferences over matchings translate to their choice behavior.

As a preparation, let us note that when there are externalities, preferences range not only over the sets of contracts that list agents as a buyer or seller but over all contracts. In this case, the alternative approach works as follows. Denote agent i's preference by $\succcurlyeq_i$ (and the strict part by $\succ_i$). We assume that $\succcurlyeq_i$ is strict if the matching for the rest of the agents is fixed, that is, if $X_{-i} \subseteq \mathcal{X}_{-i}$ is a set of contracts that do not have agent i as a buyer or seller, $X_i, X_i' \subseteq \mathcal{X}_i$ such that $X_i \neq X_i'$, then either $X_i \cup X_{-i} \succ_i X_i' \cup X_{-i}$ or $X_i' \cup X_{-i} \succ_i X_i \cup X_{-i}$. This assumption guarantees that agent i's choice function, which we construct below, is well defined. In the special case when there are no externalities, each agent's preference depends only on the set of contracts that she signs, i.e., for any $X_i, X_i' \subseteq \mathcal{X}_i$ and $X_{-i} \subseteq \mathcal{X}_{-i}$, we have $X_i \cup X_{-i} \succcurlyeq_i X_i' \cup X_{-i} \iff X_i \cup \mathcal{X}_{-i} \succcurlyeq_i X_i' \cup \mathcal{X}_{-i}$.

Example: Choice functions without prediction. We construct the choice of agent i given μ from any set X, $c_i(X|\mu) \subseteq X_i$, as follows:

$$c_i(X|\mu) \cup \mu_{-i} \succcurlyeq_i X_i' \cup \mu_{-i} \text{ for every } X_i' \subseteq X_i.$$

This is the choice behavior we assume in Example 5 of Section 4.4 and in Example 6 of Section 5.[18]

Example: Choice functions with prediction in one-to-one matching. For simplicity, we specify the choice behavior for the special case of our model in which each agent signs at most one contract. This is the one-to-one matching problem. Let $F(\{x\};\mu)$ be the set of contracts in matching μ that have to be removed for feasibility when contract x is added to matching μ. These are the contracts signed by the buyer and seller associated with contract x in matching μ. More formally,

$$F(\{x\};\mu) \equiv \mu_{b(x)} \cup \mu_{s(x)}.$$

In particular, if x is the empty contract, then $F(\{x\};\mu) \equiv \emptyset$. The choice of agent i from a set X given μ, $c_i(X|\mu) \subseteq X_i$, is then defined as follows:

$$c_i(X|\mu) \cup (\mu_{-i} \setminus F(c_i(X|\mu);\mu)) \succcurlyeq_i X_i' \cup \left(\mu_{-i} \setminus F\left(X_i';\mu\right)\right) \text{ for every } X_i' \subseteq X_i, \left|X_i'\right| \leq 1.$$

This choice behavior is implicit in many studies including Bando (2012; 2014). We can

[18] It is also consistent with the choice function constructions in Section 2. One could easily generalize this approach as follows. For each μ_{-i}, let there be a strict preference relation $\succcurlyeq_i^{\mu_{-i}}$ of agent i. The choice function can be constructed as: $c_i(X|\mu) \cup \mu_{-i} \succcurlyeq_i^{\mu_{-i}} X_i' \cup \mu_{-i}$ for every $X_i' \subseteq X_i$.

similarly construct choice functions for many-to-one matching markets or markets with feasibility constraints by appropriately changing the definition of $F(\{x\};\mu)$. In general, any deterministic theory of how agents react to the matching of an agent allows the agent to compare the resulting matchings and thus can be easily incorporated in our model.[19]

Our general approach with choice functions also sheds more light into the previous literature. Consider the case of pessimistic agents in Sasaki and Toda (1996): A blocking agent assumes that the rest of the agents are going to react so that the worst matching is going to realize afterwards. We can incorporate this pessimistic prediction about the future into the choice functions of agents and use the stability notion that we use above. In this case, an agent's choice function depends only on the set of available contracts and not on the reference matching. As a result, even though preferences may exhibit externalities choice functions do not. Hence, the existence result of Sasaki and Toda (1996) in case of pessimistic agents can be established by the classical existence result for the marriage problem of Gale and Shapley (1962) using our approach.

Our results and analysis remain the same regardless of how choice functions are constructed from agents' preferences. Furthermore, we allow for more general choice behavior including non-rationalizable ones.

3.2 Properties of Choice Functions

To guarantee the existence of stable matchings and mechanisms with desirable properties, we impose more structure on the choice functions. Let us first define the auxiliary concept of consistency.[20]

[19]In analyzing far-sighted stability based on such deterministic theories, we may need to take care of the possibility that two choices might lead to the same outcome. In such cases, the preferences over final outcomes need to be supplemented with a tie-breaking procedure to determine choice behavior. Such indifference situations never arise in the constructions 1 and 2 above. For an example of a theory of far-sighted stability based on deterministic assumptions on agents' reactions see, e.g., Acemoglu, Egorov, and Sonin (2012). Theories of far-sighted stability that are not directly based on such deterministic assumptions are harder to map into our framework; see, e.g., Konishi and Ünver (2007) and Ray and Vohra (2015).

[20]In our context, a **binary relation** $\tilde{\succeq}^\theta$ on domain $\mathcal{A}^\theta$ is a set of ordered pairs of elements from $\mathcal{A}^\theta$. It is **reflexive** if for any $\mu \in \mathcal{A}^\theta$, $\mu \tilde{\succeq}^\theta \mu$. It is **transitive**, if $\mu_1 \tilde{\succeq}^\theta \mu_2$ and $\mu_2 \tilde{\succeq}^\theta \mu_3$ imply $\mu_1 \tilde{\succeq}^\theta \mu_3$. A reflexive and transitive binary relation is called a **preorder**. In defining our conditions on choice, we set the domain of the preorder to be $\mathcal{A}^\theta = 2^{\mathcal{X}}$. Alternatively, we can restrict attention to any smaller domain that contains $\emptyset$ and satisfies $C^\theta(X|\mu) \in \mathcal{A}^\theta$ whenever $X \subseteq \mathcal{X}$ and $\mu \in \mathcal{A}^\theta$. The minimal such domain is $\mathcal{A}^\theta \equiv \bigcup_{t=0,1,\dots} \mathcal{A}^\theta_t$ where $\mathcal{A}^\theta_0 \equiv \{\emptyset\}$ and $\mathcal{A}^\theta_t$ for $t \geq 1$ are defined recursively $\mathcal{A}^\theta_t \equiv \{C^\theta(X|\mu) : X \subseteq \mathcal{X}, \mu \in \mathcal{A}^\theta_{t-1}\} \cup \mathcal{A}^\theta_{t-1}$. Since there exists a finite number of contracts, $\mathcal{A}^\theta$ is well defined; it is the set of all matchings that can be reached from the empty set by applying the choice function C^θ.

Definition 1. A preorder $\tilde{\succeq}^{\theta}$ is **consistent** with the side choice function C^{θ} if for any $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X' \supseteq X \ \& \ \mu' \tilde{\succeq}^{\theta} \mu \implies C^{\theta}\left(X'|\mu'\right) \tilde{\succeq}^{\theta} C^{\theta}\left(X|\mu\right).$$

To define our conditions, we consider consistent preorders. A consistent preorder $\tilde{\succeq}^{\theta}_1$ is **minimal** if there does not exist another consistent preorder $\tilde{\succeq}^{\theta}_2$ such that for any μ and μ', $\mu \tilde{\succeq}^{\theta}_2 \mu'$ implies $\mu \tilde{\succeq}^{\theta}_1 \mu'$. The following lemma establishes the existence and uniqueness of the minimal preorder that is consistent with a side choice function.

Lemma 1. *There exists a unique minimal preorder that is consistent with the side choice function C^{θ}.*

Proof. First of all, the preorder on the set $\mathcal{X}$ that includes all possible pairs of matchings is consistent with the choice function C^{θ}. Hence, there exists at least one preorder that is consistent with the choice function C^{θ}. Now, let us construct a minimal preorder consistent with C^{θ}. Suppose that $\{\succeq^{\theta}_1, \succeq^{\theta}_2, \ldots, \succeq^{\theta}_k\}$ is the set of all preorders that are consistent with choice function C^{θ}. Define the following binary relation: $\mu' \succeq^{\theta} \mu$ if and only if $\mu' \succeq^{\theta}_j \mu$ for every $j = 1, \ldots, k$. The binary relation $\succeq^{\theta}$ is reflexive and transitive, so it is a preorder. In addition, let $X' \supseteq X$ and $\mu' \succeq^{\theta} \mu$. Then $\mu' \succeq^{\theta}_j \mu$ for every $j = 1, \ldots, k$. By consistency of $\succeq^{\theta}_j$, we get $C^{\theta}\left(X'|\mu'\right) \succeq^{\theta}_j C^{\theta}\left(X|\mu\right)$ for every $j = 1, \ldots, k$. As a result, $C^{\theta}\left(X'|\mu'\right) \succeq^{\theta} C^{\theta}\left(X|\mu\right)$. Therefore, $\succeq^{\theta}$ is also consistent with the choice function C^{θ}. Since the number of preorders is finite, this argument shows that there exists a unique minimal preorder $\succeq^{\theta}$ which is consistent with C^{θ}. □

We define our conditions using this minimal preorder $\succeq^{\theta}$. To simplify exposition, when $\mu' \succeq^{\theta} \mu$ we say that μ' has a *better market condition* than μ for side θ.

Definition 2. Choice function C^{θ} satisfies **substitutability** if for any $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X' \supseteq X \ \& \ \mu' \succeq^{\theta} \mu \implies R^{\theta}\left(X'|\mu'\right) \supseteq R^{\theta}\left(X|\mu\right).$$

Equivalently, choice function C^{θ} satisfies **substitutability** if there exists a consistent preorder $\succeq$ such that for any $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X' \supseteq X \ \& \ \mu' \succeq \mu \implies R^{\theta}\left(X'|\mu'\right) \supseteq R^{\theta}\left(X|\mu\right).$$

Less formally, the choice function of side θ satisfies substitutability if any contract rejected from a set X given a matching μ is also rejected from a superset of X given a matching μ'

that has a better market condition than μ. When $\mu' = \mu$ or when there are no externalities, a choice function satisfies substitutability if the corresponding rejection function is monotone, or equivalently, a contract that is chosen from a larger set is also chosen from a smaller set including that contract. This special case is standard substitutability; it was introduced by Kelso and Crawford (1982) for a matching market with transfers.[21] Our definition is more general and incorporates externalities since the choice function of an agent depends on the set of contracts signed by the rest of the agents.

In substitutability, we condition the choice set and rejection set on matchings; in particular, we impose that μ' has a better market condition than μ. This is a novel property. Importantly, when there are no externalities for side θ, the preorder $\geq^\theta$ is defined as the *revealed preference* for agents on side θ.[22] In addition, substitutability reduces to the regular one studied in the literature when there are no externalities as the conditioning on matchings is no longer important. It is also satisfied in the slightly more general setting in which externalities affect agents' preferences but not their choices (for instance, if the agents' utility can be additively separated into utility from one's own contracts and utility from contracts of other agents' on the same side of the market).

Substitutability can be decomposed into two separate conditions. First is the case when $\mu' = \mu$, which is similar to the standard substitutability: we discuss this in the preceding paragraph. Second, when $X' = X$, we reject more students conditional on a matching that has a better market condition. The conjunction of these two special cases are equivalent to substitutability.

As highlighted above, if substitutability is satisfied for a preorder consistent with the choice function, then it is also satisfied for the minimal preorder $\geq^\theta$.[23] This will be useful in our applications as we do not have to find the minimal preorder consistent with the choice function but just one preorder that is consistent with the choice function. As a result, we can potentially use many preorders $\geq^\theta$ for each side θ. One example of such a preorder can be defined as follows when agents have preferences over sets of contracts: for any matchings μ and μ', $\mu' \geq^\theta \mu$ if $\mu' \geq_i \mu$ for all $i \in \theta$ (in words, side θ prefers matching μ' to μ if all agents in θ prefer

[21] See also Roth (1984), Fleiner (2003), and Hatfield and Milgrom (2005). Note that in the presence of externalities, our substitutes assumption imposes a preference restriction even on agents who sign at most one contract.

[22] When there are no externalities C^θ does not depend on the reference matching. In this case, X is revealed preferred to Y if $C^\theta(X \cup Y) = X$.

[23] In early drafts, we required that the preorder satisfies **antisymmetry**, that is, $\mu_1 \tilde{\geq}^\theta \mu_2$ and $\mu_2 \tilde{\geq}^\theta \mu_1$ imply $\mu_1 = \mu_2$; in other words, we worked with **partial order** $\geq^\theta$. This however made the substitutability notion stronger, and our results weaker. For example, the preorders that we use in Examples 1 and 3 are not partial orders, so the argument for existence of stable matchings in these examples relies on defining substitutability with respect to preorders rather than partial orders.

μ' to μ). But we are not restricting our attention to such preorders. In particular, the preorder might capture some properties of the underlying fundamentals. For instance, if agents contract over qualities and payments, we might have $\mu' \geq^{\theta} \mu$ if the profile of qualities in μ' is higher than the profile of qualities in μ (irrespective of payments, and hence of agents' utilities). In Example 1, $\mu' \geq^{\theta} \mu$ for workers if and only if the married women have better jobs in μ' compared to μ.

Next, we introduce a basic axiom for a choice function. Let us stress that this axiom is tautologically satisfied when the choice behavior is as in Section 3.1 and in our examples.

Definition 3. Choice function C^{θ} satisfies the **irrelevance of rejected contracts** if for all $X', X, \mu \subseteq \mathcal{X}$, we have

$$C^{\theta}(X'|\mu) \subseteq X \subseteq X' \implies C^{\theta}(X'|\mu) = C^{\theta}(X|\mu).$$

If choice function C^{θ} satisfies the irrelevance of rejected contracts, then excluding contracts that are not chosen does not change the chosen set. This is a basic axiom for choice functions. It has been studied in the matching with contracts literature by Aygün and Sönmez (2013) when there are no externalities. They show that, without this condition, substitutability alone does not guarantee the existence of stable matchings; but these two conditions together imply the existence. If choice functions are constructed from preferences as in Section 3.1, then the irrelevance of rejected contracts is automatically satisfied.

By construction, C^{θ} satisfies the irrelevance of rejected contracts (or substitutability) if and only if c_i satisfies the irrelevance of rejected contracts (or substitutability) for every agent i on side θ. Therefore, we can impose these two conditions on either agents' choice functions or the choice functions for each side of the market.

3.3 Examples Revisited

Now, we illustrate these properties with our examples. We focus on substitutability because it is straightforward to see that the irrelevance of rejected contracts is satisfied.

Example 1 revisited: Worker choice functions satisfy substitutability for preorder $\geq^{\theta}$ such that $\mu' \geq^{\theta} \mu$ when each married woman gets a better job in μ' compared to μ. This preorder is consistent because as there are more contracts available married women are better off since their choice functions do not exhibit externalities. The substitutes condition is satisfied because a married man becomes weakly more selective whenever his wife gets a weakly better job, so he rejects more contracts conditional on μ' compared to μ whenever $\mu' \geq^{\theta} \mu$.

Example 2 revisited: College choice functions satisfy substitutability if we define the preorder $\geq^{\theta}$ so that $\mu' \geq^{\theta} \mu$ if and only if $\max_{j \in \mu'(i)} \lambda(i, j)$ is weakly higher than $\max_{j \in \mu(i)} \lambda(i, j)$ for all colleges i.[24] This preorder is consistent with the choice functions: when more academics are around then the maximum quality of the academics a college hires goes up (whether or not the benchmark quality of academics increases). The substitutability condition is then satisfied: when more academics are around and when the benchmark quality of academics increases, each college continues to reject the academics it previously rejected.

Example 3 revisited: Using individual buyer choice functions, we can construct a choice function C^b for the buyer side.

	$\{x_1, x_2, x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot \mid \{x_1, x_2, x_3\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot \mid \{x_1, x_2\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot \mid \{x_1, x_3\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot \mid \{x_2, x_3\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot \mid \{x_1\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot \mid \{x_2\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot \mid \{x_3\})$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot \mid \emptyset)$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$

Table 2: Buyer-side choice function in Example 3.

We use the following preorder for buyers: $\{x_1, x_2\} \geq^b \{x_1, x_3\}, \{x_1\}, \{x_2\} \geq^b \{x_3\}, \emptyset$; $\{x_1, x_3\} \sim^b \{x_1\}$; and $\{x_3\} \sim^b \emptyset$. This preorder is consistent with C^b: for example, $\{x_1, x_2\} \geq^b \{x_1\}$, so we must have $C^b(\{x_1, x_3\} \mid \{x_1, x_2\}) \geq^b C^b(\{x_3\} \mid \{x_1\})$, which is true since $C^b(\{x_1, x_3\} \mid \{x_1, x_2\}) = \{x_1\} \geq^b \emptyset = C^b(\{x_3\} \mid \{x_1\})$. Likewise, $\{x_1, x_2\} \geq^b \{x_2\}$ implies $C^b(\{x_1, x_3\} \mid \{x_1, x_2\}) \geq^b C^b(\{x_1, x_3\} \mid \{x_2\})$. Again, this holds because $C^b(\{x_1, x_3\} \mid \{x_1, x_2\}) = \{x_1\} \geq^b \{x_1, x_3\} = C^b(\{x_1, x_3\} \mid \{x_2\})$. Substitutability is satisfied for this consistent preorder. For example, $\{x_1, x_2\} \geq^b \{x_1\}$, as a result, we must have $R^b(\{x_1, x_3\} \mid \{x_1, x_2\}) \supseteq R^b(\{x_3\} \mid \{x_1\})$, which is true since $R^b(\{x_1, x_3\} \mid \{x_1, x_2\}) = \{x_3\} \supseteq \{x_3\} = R^b(\{x_3\} \mid \{x_1\})$. Likewise, $\{x_1, x_2\} \geq^b \{x_2\}$ implies $R^b(\{x_1, x_3\} \mid \{x_1, x_2\}) \supseteq R^b(\{x_1, x_3\} \mid \{x_2\})$. Again, this holds because $R^b(\{x_1, x_3\} \mid \{x_1, x_2\}) = \{x_3\} \supseteq \emptyset = R^b(\{x_1, x_3\} \mid \{x_2\})$. Finally, $\{x_3\} \sim \emptyset$ implies $R^b(X \mid \{x_3\}) = R^b(X \mid \emptyset)$ for any set of contracts X, which is true.

[24] When $\mu(i)$ is empty, we set the maximum equal to $-\infty$.

4 Stable Matchings

As in classical matching theory, a key step in proving the existence of stable matchings is an algorithm akin to the deferred acceptance algorithm.

Our generalization of the deferred acceptance algorithm has two phases. First, we construct an auxiliary matching μ^* such that $C^s(\mathcal{X}|\mu^*) \preceq^s \mu^*$. Then, we use μ^* to construct a stable matching in a way resembling the classic deferred acceptance algorithm of Gale and Shapley (1962) and, particularly, its extension by Hatfield and Milgrom (2005): we run the algorithm in rounds, $t = 1, 2, \ldots$. In any round $t \geq 1$, we denote by $A^s(t)$ and $A^b(t)$ the set of contracts that are available to the buyers and sellers, respectively. Therefore, the set of contracts held at the beginning of each round is $A^s(t) \cap A^b(t)$. We also track the reference matchings for each side: $\mu^s(t)$ is the seller reference matching and $\mu^b(t)$ is the buyer reference matching.[25]

Phase 1: Construction of an auxiliary matching μ^* such that $\mu^* \succeq^s C^s(\mathcal{X}|\mu^*)$. Set $\mu_0 \equiv \emptyset$ and define recursively $\mu_k \equiv C^s(\mathcal{X}|\mu_{k-1})$ for every $k \geq 1$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$. We take the minimum m satisfying this property and set $\mu^* = \mu_m$.

We establish below that the matching constructed in phase 1 satisfies the property that $\mu^* \succeq^s C^s(\mathcal{X}|\mu^*)$.

Phase 2: Construction of a stable matching. Set $A^s(1) \equiv \mathcal{X}$ (all contracts are available to the buyers), $A^b(1) \equiv \emptyset$ (no contracts are available to the sellers), and the reference matchings are $\mu^s(1) = \mu^*$, and $\mu^b(1) = \emptyset$. In each round $t = 1, 2, \ldots$, we update these sets and matchings

[25] The tracking of reference matchings has no counterpart in earlier formulations of the deferred acceptance algorithms of, among many others, Gale and Shapley (1962), Roth (1984), Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Echenique and Yenmez (2007), Ostrovsky (2008), Hatfield and Kojima (2010), and Bando (2014). In these papers, there is no need to track reference matchings and the deferred acceptance algorithm terminates when there are no more rejections and no new offers. However, in our setting, the lack of rejections and new offers is not sufficient to stop the algorithm and we need to run it until the reference matchings converge. We run the algorithm in a symmetric way: in each round agents on both sides respond to the offers and rejections from the previous round. This is formally different from the standard approach where agents on the proposing side respond to rejections from the earlier round but the agents on the accepting side respond to offers in the current round. This difference is not substantive: we could run the deferred acceptance algorithm in the latter manner with straightforward adjustments.

as follows:

$$\begin{aligned} A^s(t+1) &\equiv \mathcal{X} \setminus R^b(A^b(t)|\mu^b(t)), \\ A^b(t+1) &\equiv \mathcal{X} \setminus R^s(A^s(t)|\mu^s(t)), \\ \mu^s(t+1) &\equiv C^s(A^s(t)|\mu^s(t)), \\ \mu^b(t+1) &\equiv C^b(A^b(t)|\mu^b(t)). \end{aligned}$$

Thus, the buyers reject some of the contracts offered in $A^b(t)$ given their reference matching $\mu^b(t)$ and the set of contracts available to the sellers after the round is $A^s(t+1) = \mathcal{X} \setminus R^b(A^b(t)|\mu^b(t))$. Likewise, the sellers reject some contracts in $A^s(t)$ conditional on their reference matching $\mu^s(t)$ and the set of contracts available to the buyers after the round is $A^b(t+1) = \mathcal{X} \setminus R^s(A^s(t)|\mu^s(t))$. We also update the reference matchings: at each round, the sellers' reference matching is the chosen set of contracts and likewise for the buyers.

We continue updating these sets until round T such that $A^s(T+1) = A^s(T)$, $A^b(T+1) = A^b(T)$, $\mu^s(T+1) = \mu^s(T)$, and $\mu^b(T+1) = \mu^b(T)$. The outcome of the algorithm is then $A^s(T) \cap A^b(T)$.

The main result of this section establishes that the algorithm terminates at some round T despite the presence of externalities and, furthermore, it produces a stable matching.

Theorem 1. *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then, the algorithm terminates, its outcome is stable, and*

$$\mu^s(T) = \mu^b(T) = A^s(T) \cap A^b(T).$$

This result implies that stable matchings exist in environments satisfying substitutability and the irrelevance of rejected contracts, e.g., in the examples of Section 2. Its proof relies on monotonicity but we need to address two complications. First, the second phase of our deferred acceptance procedure is monotonic only in some circumstances; it is the role of the first phase to guarantee monotonicity of the second phase. Second, working with preorders rather than partial orders implies that we cannot use Tarski's fixed-point theorem. Instead, we find that iterative application of phase two of the algorithm leads us to a set of matchings which are equivalent in the preorder. The relation between sets $A^s(T)$, $A^b(T)$, and reference matchings $\mu^s(T)$ and $\mu^b(T)$ allows us to then conclude that $A^s(T) \cap A^b(T)$ is a stable matching. We provide the details of the proof of the theorem in Appendix C.

In the reminder of this section, we discuss the similarities and differences with the standard deferred acceptance algorithm, consider an example of how the algorithm runs, and establish two auxiliary properties of the transformation iteratively performed in the second phase of the deferred acceptance algorithm.

4.1 An Illustration of the Deferred Acceptance Algorithm

Like the standard deferred acceptance algorithm, in each round of phase 2, substitutability and the irrelevance of rejected contracts imply that $A^s(t+1) \subseteq A^s(t)$ and $A^b(t+1) \supseteq A^b(t)$, i.e., the sellers make more offers to the buyers while more contracts are rejected by the buyers with each passing round (Lemma 2). As a consequence, the sellers' reference matching worsens and the buyers' reference matching improves. Hence, both of these two sets converges at some round t; however, the algorithm does not necessarily terminate when $A^s(t+1) = A^s(t)$ and $A^b(t+1) = A^b(t)$. Indeed, because of externalities, the set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is not necessarily stable. Instead, the algorithm converges only when $A^s(t+1) = A^s(t)$, $A^b(t+1) = A^b(t)$, $\mu^s(t+1) = \mu^s(t)$ and $\mu^b(t+1) = \mu^b(t)$. And the set of contracts held at such a round is stable.

The following example, which is a special case of Example 1, illustrates this point and shows the steps of the algorithm. This example also illustrates that our algorithm can be viewed as an ascending auction in the presence of externalities.

Example 4. Suppose there is one employer f (a firm) and two workers w_1 and w_2. The firm can sign two types of contracts with different wages: a low wage, L, and a high wage, H. The contracts are denoted as follows: $x_{1L} = (f, w_1, L)$, $x_{1H} = (f, w_1, H)$, $x_{2L} = (f, w_2, L)$, and $x_{2H} = (f, w_2, H)$. The firm would like to hire as many workers as it can and pay as low wages as it can. In other words, from any given set of contracts, the firm chooses the contract with the lowest wage associated for each worker.

Notice that in this simple example all contracts involve firm f, and hence its preferences do not depend on the reference matching (i.e., there are no externalities for the firm). Furthermore, assume that worker w_1's preferences do not depend on the reference matching (that is on what contract worker w_2 signs) and worker w_1 is willing to work only at the high wage: $x_{1H} \succ_{w_1} \emptyset \succ_{w_1} x_{1L}$. Worker w_2's preferences depend on the contract of worker w_1 (we may think of these two workers as a married couple as in Example 1). More precisely, worker w_2 is willing to work at any wage only if worker w_1 is not employed: if worker w_1 is not employed then worker w_2's preference ranking is $x_{2H} \succ_{w_2} x_{2L} \succ_{w_2} \emptyset$ and if worker w_1 is employed then worker

w_2's ranking is $\emptyset \succ_{w_2} x_{2H}, x_{2L}$. The workers' choice functions are constructed from these preferences.

Suppose that the firm plays the role of a single seller and the workers play the roles of buyers in the algorithm. The first phase of the algorithm yields $\mu^* = \{x_{1L}, x_{2L}\}$.[26] We then run the second phase as summarized in the following table.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^s(A^s(t)\|\mu^s(t))$	$C^b(A^b(t)\|\mu^b(t))$
$t = 1$	$\mathcal{X}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\emptyset$
$t = 2$	$\mathcal{X}$	$\{x_{1L}, x_{2L}\}$	$\{x_{1L}, x_{2L}\}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\{x_{2L}\}$
$t = 3$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{2L}\}$	$\{x_{1L}, x_{2L}\}$	$\{x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{2L}\}$
$t = 4$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$
$t = 5$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}\}$
$t = 6$	$\{x_{1H}, x_{2H}\}$	$\{x_{1L}, x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}\}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$
$t = 7$	$\{x_{1H}, x_{2H}\}$	$\mathcal{X}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$
$t = 8$	$\{x_{1H}\}$	$\mathcal{X}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$
$t = 9$	$\{x_{1H}\}$	$\mathcal{X}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$
$t = 10$	$\{x_{1H}\}$	$\mathcal{X}$	$\{x_{1H}\}$	$\{x_{1H}\}$		

Table 3: Steps of the Deferred Acceptance Algorithm.

In the first round, firm f chooses the low wage contract of both workers and the high wage contracts are rejected. Workers choose and reject from the initial set $A^b(1) = \emptyset$. At the end of this round, $A^s(2) = \mathcal{X}$ and $A^b(2) = \{x_{1L}, x_{2L}\}$, and the reference matchings are unchanged. In the second round, firm f faces the same choice problem while workers are now choosing from $A^b(2) = \{x_{1L}, x_{2L}\}$ and thus worker w_1 rejects the offered contract x_{1L}, while worker w_2 accepts x_{2L}.

The algorithm continues to proceed in this way. Notice that between the fourth and fifth rounds the sets of contracts available to the buyers and sellers are the same, i.e., $A^b(4) = A^b(5)$ and $A^s(4) = A^s(5)$. In the standard deferred acceptance algorithm, we could stop the algorithm here and set the outcome to the matching $A^s(4) \cap A^b(4) = \{x_{1H}, x_{2L}\}$. In our setting, this matching is not stable as w_2 prefers not to work given that w_1 is working. And, indeed, our deferred acceptance does not converge yet as the new reference matching for the workers is $\mu^b(5) = \{x_{1H}, x_{2L}\}$ which is different from $\mu^b(4) = \{x_{2L}\}$. Given this change of the reference matching, worker w_2 rejects the contract x_{2L}. The algorithm eventually converges at the ninth round and produces the matching $\{x_{1H}\}$, which is a stable matching.

[26] Since the firm's preferences do not exhibit externalities, this initial matching does not impact how the algorithm runs. However, the initial matching matters for the worker-proposing version of the algorithm.

4.2 A Characterization of Stable Matchings via Fixed Points of a Monotone Function

Let us introduce some notation for the proofs of Theorem 1 and the subsequent results. Each iteration in the second phase of our algorithm can be described as the following transformation function

$$f\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X} \backslash R^b(A^b|\mu^b), \mathcal{X} \backslash R^s(A^s|\mu^s), C^s\left(A^s|\mu^s\right), C^b(A^b|\mu^b)\right),$$

where f is a function from $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ into itself.

Function f has two important properties, monotonicity and stability of its fixed points, that are captured in the following auxiliary results. Monotonicity does not require the irrelevance of rejected contracts:

Lemma 2. *Suppose that the choice functions satisfy substitutability. Then, function f is monotone increasing with respect to the preorder $\sqsubseteq$ defined as follows:*

$$(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b) \Longleftrightarrow A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^s \tilde{\mu}^s, \mu^b \succeq^b \tilde{\mu}^b.$$

The fixed points of function f correspond to stable matchings even when choice functions do not satisfy substitutability or the irrelevance of rejected contracts:

Lemma 3. *Let $\left(A^s, A^b, \mu^s, \mu^b\right)$ be a fixed point of function f. Then $A^s \cup A^b = \mathcal{X}$ and*

$$\mu^s = \mu^b = A^s \cap A^b = C^b(A^b|\mu^b) = C^s(A^s|\mu^s).$$

The straightforward proofs of these two lemmas are provided in Appendix C.

When choice functions satisfy substitutability and the irrelevance of rejected contracts, a matching is stable if and only if it can be supported as a fixed point of f.

Theorem 2. *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then a matching μ is stable if and only if there exist sets of contracts $A^s, A^b \subseteq \mathcal{X}$ such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of function f.*

The proof is provided in Appendix C.

4.3 Comments

The proof of Theorem 1 does not rely on Tarski's fixed point theorem, which is routinely used in the matching literature (e.g., see Adachi, 2000). In fact, Tarski's fixed point theorem cannot be directly applied in our setting because even though f is monotone increasing, the domain of f does not have to be a (complete) lattice. In addition, there do not have to exist matchings that are optimal for buyers or sellers. As a result, the standard approach of applying f to the extreme points to get a monotone sequence that converges to a fixed point fails. Furthermore, the binary relation $\sqsubseteq$ on the domain of f is not a partial order, which means that even if a monotone sequence exists it would not necessarily converge to a fixed point as the preorder $\sqsubseteq$ could cycle.

Theorem 1 establishes that stable matchings exist when choice functions satisfy substitutability and the irrelevance of rejected contracts. Both conditions are necessary in the sense that when only one of them is satisfied there may not be any stable matchings: Example 1 of Aygün and Sönmez (2013) satisfies substitutability for the revealed preference but there exists no stable matching (because the irrelevance of rejected contracts fails). In the next example, the irrelevance of rejected contracts is satisfied but there exists no stable matching.

Example 5. Suppose that there are two buyers b_1, b_2 and one seller, s_1. There is only one contract associated with every seller-buyer pair. Let the contract between b_1 and s_1 be x_1 and the contract between b_2 and s_1 be x_2. Since there is only one seller, there are only externalities for buyers. Agents have the following preferences:

$$
\begin{aligned}
\succ_{b_1} \quad &: \quad \{x_1\} \succ \emptyset \text{ and } \{x_2\} \succ \{x_1, x_2\}; \\
\succ_{b_2} \quad &: \quad \{x_1, x_2\} \succ \{x_1\} \text{ and } \emptyset \succ \{x_2\}; \\
\succ_{s_1} \quad &: \quad \{x_1, x_2\} \succ \{x_1\} \succ \{x_2\} \succ \emptyset.
\end{aligned}
$$

Construct agents' choice functions from their preferences. As a result, the choice functions satisfy the irrelevance of rejected contracts. Yet there exists no stable matching. To see this, first note that $\emptyset$ is not a stable matching because (b_1, s_1) forms a blocking pair with contract x_1. Second, $\{x_1\}$ is not a stable matching because (b_2, s_1) forms a blocking pair with contract x_2. Third, $\{x_2\}$ is not a stable matching because it is not individually rational for buyer b_2. Finally, $\{x_1, x_2\}$ is not a stable matching because it is not individually rational for buyer b_1.

There exists no stable matchings because substitutability fails. Recall that the minimal unique preorder $\succeq^b$ exists (Lemma 1). By consistency $C^b(x_1|\emptyset) \succeq^b C^b(\emptyset|\emptyset)$, so $x_1 \succeq^b \emptyset$. A further application of consistency gives $x_2 = C^b(x_2|x_1) \succeq^b C^b(\emptyset|\emptyset) = \emptyset$. Finally, $\emptyset = C^b(x_1|x_2) \succeq^b C^b(x_1|\emptyset) = x_1$. Therefore, we get that $x_1 \sim^b \emptyset$. However, substitutability implies that $\emptyset = R^b(x_2|x_1) = R^b(x_2|\emptyset) = x_2$, which cannot hold. Thus, substitutability fails in this example.

5 Pareto Efficiency and Side-Optimal Stable Matchings

Two key normative properties in the standard theory of stable matchings is Pareto efficiency of stable matchings and the existence of side-optimal stable matchings. Pareto efficiency extends to our setting as follows:

Theorem 3. *Suppose that the choice functions satisfy substitutability. If matching μ is stable then it is Pareto efficient in the following sense: there is no other matching $\nu \neq \mu$ such that $\nu = c_i(\nu \cup \mu|\mu)$ for every agent i.*

The argument resembles a similar argument in the no-externalities case. We prove a stronger result in Appendix B (Proposition 1).

The counterpart of the side-optimal stable matchings in the setting with externalities is more subtle and it is given by the following result. Before stating this result, we define the following concepts.

Definition 4. A stable matching μ is θ-**optimal** if $\mu \succeq^\theta \mu'$ for every stable matching μ', it is θ-**pessimal** if $\mu \preceq^\theta \mu'$ for every stable matching μ'.

In the standard stable matching theory without externalities, side optimality is measured with respect to the revealed preference of agents on this side (e.g. Roth, 1984). This standard result is subsumed.

Theorem 4. *Suppose that the choice functions satisfy substitutability, the irrelevance of rejected contracts, and, in addition, for side θ there exists a matching $\bar{\mu}^\theta$ such that for any matching μ, $\bar{\mu}^\theta \succeq^\theta \mu$. Then, there exists a θ-optimal stable matching $\hat{\mu}$, which is also a $-\theta$-pessimal stable matching.*

Proof. Without loss of generality assume that $\theta = s$. For any $(A^s, A^b, \mu^s, \mu^b) \in 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ we have $(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq (A^s, A^b, \mu^s, \mu^b)$. Therefore, $(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq f(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$. By Lemma 2, function f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^{k-1}(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq f^k(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$ for every k. Since $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is a finite set, this sequence converges at some point as in the proof of Theorem 1, so there exists k such that $f^{k-1}(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset) = f^k(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset)$. Therefore, $f^{k-1}(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset)$ is a fixed point of f. By Lemma 3 there is $\left(\hat{A}^s,\hat{A}^b,\hat{\mu},\hat{\mu}\right)$ that is equal to $f^{k-1}(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset)$. Theorem 2 tells us that $\hat{\mu}$ is a stable matching.

We next show that $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. Let μ be any stable matching. By Theorem 2, there exist A^s and A^b such that (A^s,A^b,μ,μ) is a fixed point of f. Since $(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq (A^s,A^b,\mu,\mu)$ and f is monotonic increasing, f can be applied repeatedly while preserving the order. Therefore, $f^k(\mathcal{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq f^k(A^s,A^b,\mu,\mu)$ for every k, which implies $\left(\hat{A}^s,\hat{A}^b,\hat{\mu},\hat{\mu}\right) \sqsupseteq (A^s,A^b,\mu,\mu)$. Therefore, $\hat{\mu} \succeq^s \mu$ and $\hat{\mu} \preceq^b \mu$, so $\hat{\mu}$ is the seller-optimal and buyer-pessimal stable matching. □

The assumption that there exists a matching $\bar{\mu}^\theta$ such that for any matching μ, $\bar{\mu}^\theta \succeq^\theta \mu$ plays a crucial role in the proof of Theorem 4. It is not innocuous but it is satisfied in all the examples of Sections 2 and 7. In the absence of externalities, this assumption is automatically satisfied when $\succeq^\theta$ is defined as $\mu \succeq^\theta \mu'$ if and only if for every $i \in \theta$, $c_i(\mu(i) \cup \mu'(i)) = \mu(i)$ (or, if and only, if all agents on side θ prefer μ over μ'). Indeed, we can take $\bar{\mu}$ to be the set of contracts that assigns each agent on side θ his unconstrained best set of contracts.[27] Furthermore, for this preorder $\succeq^\theta$ substitutability and irrelevance of rejected contracts are equivalent to the standard ones without externalities. Thus, Theorem 4 subsumes the standard insight that, in the absence of externalities, there exists a θ-optimal stable matching with respect to $\succeq^\theta$ if preferences satisfy substitutability and the irrelevance of rejected contracts. This matching is also $(-\theta)$-pessimal.

Furthermore, our assumption on $\bar{\mu}$ is equivalent to the following: for any two matchings μ and μ', there exists another matching $\tilde{\mu}$ such that $\tilde{\mu} \succeq^\theta \mu$ and $\tilde{\mu} \succeq^\theta \mu'$. In fact, in light of our analysis of the algorithm that we provide above, it is enough to impose this assumption on matchings μ such that $C^\theta(\mathcal{X}|\mu) \preceq^\theta \mu$.

Before we end the discussion on side-optimal stable matchings, we provide an example which shows that the assumption that there exists a side-optimal matching is necessary for Theorem 4.

Example 6. Modify Example 5 with the following preferences:

[27]Notice that this point remains true regardless of whether all sets of contracts are matchings or only some sets of contracts are matchings because of some feasibility constraints as, for instance, in one-to-one matching. This is so because we allow $\bar{\mu}$ to be any set of contracts.

$$\succ_{b_1} \;:\; \emptyset \succ \{x_1\} \text{ and } \{x_1, x_2\} \succ \{x_2\};$$
$$\succ_{b_2} \;:\; \emptyset \succ \{x_2\} \text{ and } \{x_1, x_2\} \succ \{x_1\};$$
$$\succ_{s_1} \;:\; \{x_1, x_2\} \succ \{x_1\} \succ \{x_2\} \succ \emptyset.$$

Construct agents' choice functions from their preferences. As a result, the choice functions satisfy the irrelevance of rejected contracts. Furthermore, the standard substitutability is satisfied for the seller choice function. For buyers, consider the preorder $\succeq^b$ such that $\emptyset \succeq^b \emptyset$ and no other matchings are comparable. This preorder is consistent because conditional on the empty set both buyers do not choose any contract. In addition, the buyer-side choice function satisfies substitutability because the buyer-side rejection function is monotone conditional on the empty set.

There exists no buyer-optimal stable matching in this example because both the empty set and $\{x_1, x_2\}$ are stable matchings which cannot be compared by the preorder $\succeq^b$. The reason is that there exists no buyer-optimal matching μ^b such that $\mu^b \succeq^b \mu$ for all matchings μ, which is the additional assumption needed for the existence of side-optimal stable matchings on top of substitutability and irrelevance of rejected students.

6 Comparative Statics and "Vacancy Chain" Dynamics

In this section, we first present a comparative statics result that goes beyond the classic theory of stable matchings. Then we look at the welfare implications of an agent retiring from the market.

6.1 Comparative Statics on Strength of Externalities and Substitutes

How do stable matchings change when externalities and substitutability are strengthened? To answer this question, we first introduce the notions of having weaker externalities and stronger substitutability.

Definition 5. Choice function $\hat{C}^\theta$ exhibits **stronger substitutability** than choice function C^θ if $R^\theta(X|\mu) \subseteq \hat{R}^\theta(X|\mu)$ for any $\mu, X \subseteq \mathcal{X}$.

Strengthening the substitutes means that agents reject more contracts. Equivalently, we can think of shrinking the choice function so that agents choose only a subset of the previously chosen contracts.[28] To get a sense of this assumption, consider for instance Example 2 (in its general, quantile form). In this example, the larger k is the stronger substitutability of the colleges' choice function.[29]

Definition 6. Choice function $\hat{C}^\theta$ exhibits **weaker externalities** than choice function C^θ if $\hat{C}^\theta(X|\mu) \hat{\gtrsim}^\theta C^\theta(X|\mu)$ for any $\mu, X \subseteq \mathcal{X}$.

Here $\hat{\gtrsim}^\theta$ is a consistent preorder for choice function $\hat{C}^\theta$ (not necessarily the unique minimal one). Note that if choice function $\hat{C}^\theta$ exhibits no externalities then it has weaker externalities than any other choice function when $\hat{\gtrsim}^\theta$ is the revealed preference for side θ. In the context of Example 2, the externalities are weaker when the benchmark ratio k is higher. Notice that the choice function when $k = 1$ and the choice function when $k = 0$ exhibit no externalities, and thus have weaker externalities than the intermediate choice functions.

In the result below, we consider two seller choice functions C^s and $\hat{C}^s$. Suppose that preorder $\gtrsim^s$ is consistent with C^s and preorder $\hat{\gtrsim}^s$ is consistent with $\hat{C}^s$. Assume that both choice functions satisfy the irrelevance of rejected contracts and substitutability.

Theorem 5. *Suppose that $\hat{C}^s$ exhibits stronger substitutability and weaker externalities than C^s. Then for any (C^b, C^s)-stable matching μ there exists a $(C^b, \hat{C}^s)$- stable matching μ^* such that $\mu \geq^b \mu^*$ and $\mu^* \hat{\gtrsim}^s \mu$.*

Proof. For any $A^s, A^b, \mu^s, \mu^b \subseteq \mathcal{X}$, let

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X} \backslash R^b(A^b|\mu^b), \mathcal{X} \backslash \hat{R}^s(A^s|\mu^s), \hat{C}^s\left(A^s|\mu^s\right), C^b(A^b|\mu^b)\right).$$

Since μ is a (C^b, C^s)-stable matching, there exist $A^s, A^b \subseteq \mathcal{X}$ such that (A^s, A^b, μ, μ) is a fixed point of f (Theorem 2). By Lemma 3, $C^s(A^s|\mu) = C^b(A^b|\mu) = \mu$. By strong substitutes, $X \setminus \hat{R}^s(A^s|\mu) \subseteq X \setminus R^s(A^s|\mu)$; by weaker externalities, $\hat{C}^s\left(A^s|\mu\right) \hat{\gtrsim}^s C^s(A^s|\mu)$. Hence, $(A^s, A^b, \mu, \mu) = f(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}(A^s, A^b, \mu, \mu)$. Since $\hat{f}$ is monotone $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}^k(A^s, A^b, \mu, \mu)$ for all $k \geq 1$. Since the number of contracts is finite, there exists k such that $\hat{f}^{k-1}(A^s, A^b, \mu, \mu)$ is a fixed point of $\hat{f}$ as in the proof of Theorem 1. By Lemma 3, $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) = (\hat{A}^s, \hat{A}^b, \mu^*, \mu^*)$, and by Theorem 2, μ^* is a $(C^b, \hat{C}^s)$-stable matching. By construction, $\mu^* \hat{\gtrsim}^s \mu$ and $\mu \geq^b \mu^*$. $\square$

[28] In the terminology of Echenique and Yenmez (2015), choice function C^θ is an expansion of choice function $\hat{C}^\theta$ if for any $\mu, X \subseteq \mathcal{X}$, $C^\theta(X|\mu) \supseteq \hat{C}^\theta(X|\mu)$. This is equivalent to the stronger substitutes comparison above. Note that the result of this subsection specialized to the setting without externalities does not have a counterpart in Echenique and Yenmez (2015).

[29] In Example 8, which is in Section 7, the choice functions satisfy stronger substitutability as an attorney's profits from contracts signed by the attorney decrease relative to his profits from working on contracts signed by other attorneys.

In the context of Example 2, as colleges raise the hiring benchmark, the quality of academics hired in stable matchings increases. Whenever the side-optimal and side-pessimal stable matchings exist, the market conditions are better for buyers in the buyer-optimal $\hat{\geq}$-stable matching than in the buyer-optimal $\geq$-stable matching; and the converse holds for the sellers.

Example 1 Revisited: Let us consider the local labor market of Example 1. There are no externalities for the firms and their choice functions satisfy substitutability and the irrelevance of rejected contracts. Initially, some agents are married while some are not. Consider two workers: woman w and man m who are both single. Since they are single, there are no externalities for them and they have preferences over acceptable firms. Let C^s denote the choice function of the workers (so they act as sellers in the previous definitions). Suppose that woman w and man m get married. Woman w still ranks the firms in the same way. However, as woman w gets a better firm man m ranks fewer firms in the same order. There are no other changes in the market. Let $\hat{C}^s$ be the new choice function of the workers.

For the sake of the discussion, we use a slightly different preorder than the one we have studied in Section 3.3: let $\mu' \hat{\geq}^s \mu$ whenever all women have weakly better firms in μ' than μ. Like before, it is easy to see that $\hat{\geq}^s$ is a consistent preorder and that choice function $\hat{C}^s$ satisfies substitutability for this preorder. Furthermore, choice function $\hat{C}^s$ exhibits stronger externalities than choice function C^s because man m rejects more firms when he gets married and other workers have the same choice functions. Likewise, choice function $\hat{C}^s$ exhibits weaker externalities than choice function C^s because married women have the same preferences over workers under both scenarios. As a result, Theorem 5 implies that when agents get married in a local labor market firms are worse off (in the revealed preference sense) and all women get weakly better firms.

When one side of the market faces no externalities, then the preorder $\geq^\theta$ that ranks μ above μ^* whenever all agents on this side prefer μ over μ^* is consistent with this side's choice behavior. Hence, if, say, buyers face no externalities then they would all prefer μ over μ^*. This gives us the following.

Corollary 1. *Suppose that $\hat{C}^s$ does not exhibit any externalities and that $\hat{C}^s$ has stronger substitutes than C^s. Then for any (C^b, C^s)-stable matching μ there exists a $(C^b, \hat{C}^s)$- stable matching μ^* such that all buyers prefer μ over μ^*.*

Remark 1. Both our conditions, stronger substitutability and weaker externalities, can be weakened by assuming that they hold only when $C^\theta(X|\mu) = \mu$. The weaker assumptions suffice since in the proof we apply these conditions to C^s and $\hat{C}^s$ only when $C^s(A^s|\mu) = \mu$.

6.2 Vacancy Chain Dynamics

Let us consider the classic retirement problem in matching. Suppose that agent $i \in \theta$ retires. Then all of the contracts that agent i has signed are annulled. Some agents may be affected by the removal of these contracts. Therefore, agents may want to add new contracts, or they may want to remove some of the existing contracts. But the addition or removal of a new contract may also affect the remaining agents in the market, which may lead to other changes in the set of contracts. We analyze such changes and show that there is a *vacancy chain dynamics* (Crawford, 1991; Blum, Roth, and Rothblum, 1997) that leads to a stable matching in which agents on side θ are better off and agents on side $-\theta$ are worse off. Similar vacancy chain dynamics have been studied in different matching markets without externalities (e.g., Kelso and Crawford, 1982; Hatfield and Milgrom, 2005). Our construction shows that vacancy chain dynamics extend to the setting with externalities.

Without loss of generality, we fix the choice functions of agents other than some seller i while we compare two possible choice functions of seller i, say c_i and $\hat{c}_i$, where this agent rejects all contracts under $\hat{c}_i$. Let the corresponding rejection functions be r_i and $\hat{r}_i$, respectively. Less formally, the retirement of seller i is interpreted as no offers being accepted by seller i and so all offers being rejected by her. On the other hand, the rejection set for the buyers is the same. For any $X, \mu \subseteq \mathcal{X}$, $\hat{C}^s(X|\mu) \equiv \hat{c}_i(X|\mu) \cup \bigcup_{j \in s\setminus\{i\}} c_j(X|\mu)$.

We assume that C^s satisfies substitutability and the irrelevance of rejected contracts for preorder $\succeq^s$. In addition, assume that $\hat{C}^s$ satisfies substitutability and the irrelevance of rejected contracts for preorder $\hat{\succeq}^s$. Likewise, C^b satisfies substitutability and the irrelevance of rejected contracts for preorder $\succeq^b$. Notice that in the contexts of our motivating examples, all these assumptions are satisfied.

To study the vacancy-chain dynamics, we need to modify the function f as in the proof of Theorem 5. For any $A^s, A^b, \mu^s, \mu^b \subseteq \mathcal{X}$,

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X}\setminus R^b(A^b|\mu^b), \mathcal{X}\setminus \hat{R}^s(A^s|\mu^s), \hat{C}^s\left(A^s|\mu^s\right), C^b(A^b|\mu^b)\right).$$

Let $(A^s(0), A^b(0), \mu^s(0), \mu^b(0))$ be the initial matching that is stable with seller i present in the market. After removing seller i from the market, agents start recontracting dynamically. This process can be described through the function $\hat{f}$. Let $(A^s(t), A^b(t), \mu^s(t), \mu^b(t)) \equiv \hat{f}(A^s(t-1), A^b(t-1), \mu^s(t-1), \mu^b(t-1))$. We call this the **vacancy chain dynamics**. In our setting, $\hat{f}$ is monotonic since we impose the substitutes and irrelevance of rejected contracts assumptions both on the original choice function profile and on the profile when agent i rejects all offers (or, equivalently, has retired).

Theorem 6. *Suppose that $\hat{C}^s$ exhibits weaker externalities than C^s. Let (A^s, A^b) be a (C^s, C^b)-stable set of contracts with stable matching $\mu \equiv A^s \cap A^b$. Then the vacancy chain dynamics starting at (A^s, A^b, μ, μ) converges to $(A^*_s, A^*_b, \mu^*, \mu^*)$ where μ^* is a $(\hat{C}^s, C^b)$-stable matching such that $\mu^* \hat{\succeq}^s \mu$ and $\mu \succeq^b \mu^*$.*

The assumption that $\hat{C}^s$ exhibits weaker externalities than C^s is satisfied in Example 2. Thus, in this example the closure of one of the colleges leads to an increase in the quality of academics hired by the remaining colleges.

Proof. Since (A^s, A^b) is a stable set of contracts, (A^s, A^b, μ, μ) is a fixed point of f (Theorem 2). By Lemma 3, $C^s(A^s|\mu) = C^b(A^b|\mu) = \mu$. By definition, $\mathcal{X} \setminus \hat{R}^s(A^s|\mu) \subseteq \mathcal{X} \setminus R^s(A^s|\mu)$. By weaker externalities, we have $\hat{C}^s\ (A^s|\mu) \hat{\succeq}^s C^s(A^s|\mu) = \mu$. Hence, $(A^s, A^b, \mu, \mu) = f(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}(A^s, A^b, \mu, \mu)$. Since $\hat{f}$ is monotone $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}^k(A^s, A^b, \mu, \mu)$ for all $k \geq 1$. Since the number of contracts is finite, there exists k such that $\hat{f}^{k-1}(A^s, A^b, \mu, \mu)$ is a fixed point of $\hat{f}$ as in the proof of Theorem 1. By Lemma 3, $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) = (\hat{A}^s, \hat{A}^b, \mu^*, \mu^*)$, and by Theorem 2, μ^* is a stable matching in the market without seller i. By construction, $\mu^* \hat{\succeq}^s \mu$ and $\mu \succeq^b \mu^*$. □

7 Additional Examples

In this section, we provide additional examples that satisfy substitutability.

7.1 Dynamic matching

Example 7. [Dynamic Matching] Firms and workers arrive to a two-sided matching market at times $t = 1, \ldots, T$. Workers who arrive at time t can wait and match at any time $t, t+1, \ldots, T$. At each time t a unique firm f_t arrives and either matches with one of the workers that is available at this time, or leaves unmatched. Firm f_t's ranking of workers is exogenously fixed but this firm's set of acceptable workers depends on the matches of firms $f_1, \ldots, f_{t-1}$: the higher firm f_1's worker in f_1's ranking, the more selective firm f_t becomes. If firm f_1 hires the same worker in two matchings, then the higher firm f_2's worker in f_2's ranking, the more selective firm f_t becomes, etc., lexicographically.

In this example, a consistent preorder for the firms is defined as follows: $\mu' \geq^{\theta} \mu$ if and only if for some firm f we have $\mu'(f) >_f \mu(f)$ and $\mu'(f') \geq_{f'} \mu(f')$ for all firms f' matched before f. This preorder is consistent with the choice functions, and the substitutability condition is satisfied as choosing out of larger (in inclusion sense) choice set conditional on a matching higher in this preorder, each firm continues to reject the worker it previously rejected.[30]

7.2 Sharing

Our theory applies to situations in which agents share profits, for instance because they work for the same firm, or have some insurance arrangements, or benefit from a public good financed by taxes on their private income. The following example illustrates a situation in which there is profit sharing.

Example 8. [Profit Sharing] Agents on one side of the market represent attorneys organized in law firms. Each attorney can work on up to $k \geq 0$ contracts with clients on the other side of the market; an attorney works on all contracts he or she signs and the attorney can also work on selected contracts signed by others in the same firm. Each contract allows an arbitrary number of attorneys to contribute; the profit an attorney makes from a contract does not depend on how many other attorneys contribute to it.[31] Each attorney prioritizes the contracts she works on, and the profit attorney i earns on a contract depends on whether it is the first, second, etc. contract in attorney i's priorities. We assume that each attorney must prioritize the contracts she signs over other contracts that she works on.

Attorneys choose what contracts to sign and what contracts to work on so as to maximize their profits: An attorney's profit is the sum of the profits from all the contracts she works on whether she signed it or not. We denote by $\lambda(x, i, \ell) \geq 0$ the profit that accrues to attorney i from working on contract x that she prioritizes in position $\ell \in \{1, \ldots, k\}$. For simplicity, let us also assume that there are no indifferences.This example satisfies our assumptions provided $\lambda(x, i, 1) > \lambda(y, i, \ell)$ for all contracts x and y as long as attorney i is the signatory of contract x and $\ell > 1$.

Attorney choice functions satisfy substitutability if we define the preorder $\geq^{\theta}$ so that $\mu' \geq^{\theta} \mu$ if and only if $\max_{x \in \mu'(i)} \lambda(x, i, 1) \geq \max_{x \in \mu(i)} \lambda(x, i, 1)$ for all agents $i \in \theta$.[32] This preorder is consistent with choice: When more contracts are available, the profitability of the best contract

[30] We would like to thank Maciej Kotowski for suggesting this example.
[31] This assumption and some of our other assumptions can be relaxed.
[32] We use the convention that the maximum over the empty set is $-\infty$.

signed by each attorney goes up (irrespective of what contracts other attorneys sign). The substitutability condition holds for each attorney i: When more contracts are available and when the profitability of the best contract signed by other attorneys (and hence the outside option of attorney i) increases, the attorney continues to reject the contracts she previously rejected.

7.3 Interoperability

Our theory also applies to situations in which agents choose basic products with no regard to the choices of others but choose add-ons in a way that depends on others' choices of basic products. For instance, consider buyers who choose between Mac, PC, and Linux computers (and operating systems) in a way that does not depend on other buyers' choices and who take the hardware/operating system choices of others into account when buying productivity software.

Example 9. [Interoperability and Add-on Contracts] Suppose agents on one side (buyers) sign two types of contracts with sellers on the other side: for instance, agents might be signing primary contracts and add-on (or maintenance) contracts. These two classes of contracts are disjoint.[33] In line with the literature on add-on pricing, suppose that agents ignore the add-on contracts when deciding which primary contracts to sign (Gabaix and Laibson, 2006), and suppose that each agent signs at most one primary contract and that there are no externalities among primary contracts.[34]

We assume that no agent's choice of add-on contracts depends on the other agents' choices of add-on contracts, and we allow a buyer's choice among add-on contracts to depend on his and the other agents' choices of primary contracts in an arbitrary way as long as the buyer rejects weakly more (in the inclusion sense) add-on contracts out of X conditional on μ than he would reject out of X' conditional on μ' whenever $X \supseteq X'$ and the agent prefers his primary contracts in μ to those in μ'.

Buyer choice functions satisfy substitutability for the preorder $\geq^\theta$ such that $\mu' \geq^\theta \mu$ when

[33] Similar examples can be written for hardware contracts and software contracts, or contracts on inputs and outputs.

[34] Formally, we assume that each buyer's choice among primary contracts does not depend on other agents' matches nor on the availability of add-on contracts. One reason that the agents ignore add-on contracts when signing primary contracts might be that the agents do not know which add-on contracts are available when signing the primary contracts as in Ellison (2005). We can relax the assumption that each agent signs at most one primary contract and assume instead that each agent's choice among primary contracts satisfies the standard substitutes assumption (see the next section).

each buyer prefers her primary contracts signed under μ' to those signed under μ. This preorder is consistent: $\geq^{\theta}$ depends only on primary contracts, and each agent prefers to choose from larger choice sets over choosing from smaller choice sets. It is enough to check substitutability separately for the primary contracts and the add-on contracts: it holds for the primary contracts as the choice over them is not affected by externalities, and it holds for the add-on contracts as we explicitly assumed it.

8 Conclusion

In this paper, we have studied a two-sided matching problem with externalities where each agent's choice depends on other agents' contracts. For such settings, we have developed the theory of stable matchings by introducing a new substitutability condition when externalities are present. More explicitly, we have studied the existence of stable matchings, Pareto efficiency of stable matchings, side-optimal stable matchings, vacancy-chain dynamics, the deferred acceptance algorithm, comparative statics depending on the strength of externalities and substitutes, and the rural hospitals theorem (which is in Appendix A). Unlike the previous matching literature, we have not relied on fixed point theorems; instead, we have used elementary techniques to overcome the difficulties associated with externalities.

We believe that our notion of substitutability will be useful to study other important questions in matching markets with externalities. For example, the relation between pairwise-stability stability, group-stability, core, and other stability concepts has been an important question in classical matching theory at least since Blair (1988). We analyze the relation between pairwise and group stability in Appendix B, but many related questions remain open. The strategy-proofness of deferred acceptance algorithm (for the proposing side) has been another important question extensively studied since Dubins and Freedman (1981). We think that a deferred acceptance procedure remains strategy-proof in our setting provided we impose the Law of Aggregate Demand a la Hatfield and Milgrom (2005); we leave an exploration of this question for future work. Finally, even though we have studied two-sided markets, we think that our techniques are applicable to more general markets such as the supply chain networks of Ostrovsky (2008) where externalities may naturally appear.

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Appendix A: Law of Aggregate Demand and the Rural Hospitals Theorem

We provide a generalization of the law of aggregate demand (Hatfield and Milgrom, 2005) and size monotonicity (Alkan and Gale, 2003). In markets without externalities, this generalization is due to Fleiner (2003). For each contract $x \in \mathcal{X}$, there is a corresponding weight denoted by $w(x)$, which is strictly positive. The generalized law of aggregate demand requires that for agent $i \in \theta$ the total weight of contracts chosen from X conditional on μ is weakly smaller than the total weight of contracts chosen from X' conditional on μ' for any $X' \supseteq X$ and $\mu' \geq^\theta \mu$. For a set of contracts $X \subseteq \mathcal{X}$, let $w(X) \equiv \sum_{x \in X} w(x)$. We provide a formal definition as follows.

Definition 7. Choice function c_i satisfies **the law of aggregate demand** if $i \in \theta$ and for any $X \subseteq X'$ and $\mu \leq^\theta \mu'$ then $w(c_i(X|\mu)) \leq w(c_i(X'|\mu'))$.

Previous definitions in the matching literature are restricted to the settings without externalities, and assume that the weight on all contracts are exactly equal (with the only exception of Fleiner (2003)). Under this assumption, the generalized law of aggregate demand reduces to for any $X \subseteq X'$ and $\mu \subseteq \mathcal{X}$, $|c_i(X|\mu)| \leq |c_i(X'|\mu)|$. In terms of the demand metaphor of Hatfield and Milgrom (2005), all contracts are traded at price one. In contrast, we allow any prices.

We study how the weight of contracts changes for an agent in different stable matchings. We show that the weight remains the same regardless of the stable matching. This extends the rural hospitals theorem of Roth (1986) in two directions: We allow different contracts to have different weights and also preferences of an agent can depend on contracts signed by others.

Theorem 7. *Suppose that choice functions satisfy substitutability, the law of aggregate demand for a weight function w, and that there exists a matching $\bar{\mu}^\theta$ such that for any $\mu \in \mathcal{M}^\theta$, $\bar{\mu}^\theta \geq^\theta \mu$ for side θ. Then, for any two stable matchings μ and μ', $w(\mu_i) = w(\mu'_i)$ for every agent i.*

Proof. First let us observe that since all weights are strictly positive, substitutability and the law of aggregate demand imply the irrelevance of rejected contracts. This is easy to see: Suppose that $X', X, \mu \subseteq \mathcal{X}$ are such that $c_i(X'_i|\mu) \subseteq X_i \subseteq X'_i$ for agent i. Then substitutability implies that $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$. Since weights are positive we get $w(c_i(X_i|\mu)) \geq w(c_i(X'_i|\mu))$. Now, since $X_i \subseteq X'_i$, the law of aggregate demand implies that $w(c_i(X_i|\mu)) \leq w(c_i(X'_i|\mu))$. Consequently, we need to have $w(c_i(X_i|\mu)) = w(c_i(X'_i|\mu))$. Since all weights are strictly positive and $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$, we get $c_i(X_i|\mu) = c_i(X'_i|\mu)$, the desired conclusion.

Without loss of generality assume that $\theta = s$. Then, by Theorem 4, there exists a stable matching μ^*, which is seller-optimal and buyer-pessimal simultaneously. We show that for any stable matching μ, $w(\mu_i) = w(\mu^*_i)$. As it is shown in the proof of Theorem 4, f has two fixed points $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) such that $(A^{*s}, A^{*b}, \mu^*, \mu^*) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $A^{*s} \supseteq A^s$, $A^{*b} \subseteq A^b$, $\mu^* \geq^s \mu$ and $\mu^* \leq^b \mu$. Now by the law of aggregate demand for any $i \in S$, $w(c_i(A^{*s}|\mu^*)) \geq w(c_i(A^s|\mu))$, which is equivalent to $w(\mu^*_i) \geq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When this is summed over all sellers, we get $w(\mu^*) \geq w(\mu)$. Similarly, for any $i \in B$, $w(c_i(A^{*b}|\mu^*)) \leq w(c_i(A^b|\mu))$, which is equivalent to $w(\mu^*_i) \leq$

$w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When summed over all buyers, this implies $w(\mu^*) \leq w(\mu)$. Therefore, $w(\mu^*) = w(\mu)$, moreover, all of the individual inequalities must hold as equalities implying that for any agent i, $w(\mu_i^*) = w(\mu_i)$. □

Remark 2. The first part of the proof shows that the law of aggregate demand and the substitute condition imply the irrelevance of rejected contracts, thus extending an analogous result in Aygün and Sönmez (2013) to the setting with externalities. This part of the proof relies on the weights being strictly positive; the remainder of the proof does not. In particular, our proof thus establishes that the analogue of the rural hospitals theorem holds true for any profile of real weights, not necessarily positive, as long as we assume that that the choice functions satisfy the irrelevance of rejected contracts. In addition, under the assumptions of the theorem, an agent's choice from the same set conditional on two ranked matchings needs to be the same. Indeed, let $i \in \theta$ be an agent. Suppose that $X, \mu, \mu' \subseteq \mathcal{X}$ are such that $\mu \preceq^{\theta} \mu'$. Then, by substitutability, $c_i(X|\mu) \supseteq c_i(X|\mu')$. But the law of aggregate demand implies that $w(c_i(X|\mu)) \leq w(c_i(X|\mu'))$. Since all weights are positive, we get that $c_i(X|\mu) = c_i(X|\mu')$. This argument does not mean that we cannot have externalities because the choice conditional on two matchings that are not ranked with respect to $\succeq^{\theta}$ can still be different.

Appendix B: Group Stability

A set $X \subseteq \mathcal{X}$ **blocks** matching μ if $X \nsubseteq \mu$ and for all $i \in \mathcal{I}$ we have $X_i \subseteq c_i(\mu \cup X|\mu)$. Less formally, conditional on matching μ, every agent who is associated with a contract in X wants to sign all contracts in X associated with her. In this case, X is also called a **blocking set** for μ. A matching is **group stable** if it is individually rational matching and there is no blocking set of contracts. Without externalities, this stability concept has been used before (see, e.g., Roth, 1984 and Hatfield and Kominers (2016)).

Proposition 1. ***[Equivalence of Stability and Group Stability]*** *Suppose that choice functions satisfy substitutability. Then a matching is stable if and only if it is group stable.*

See Roth and Sotomayor (1990); Echenique and Oviedo (2006); Hatfield and Kominers (2016) for earlier developments of this equivalence when there are no externalities. In particular, Hatfield and Kominers (2016) prove the same result when there are no externalities. The same proof works in our setting as well. More precisely, the following lemma is enough to prove the proposition, which does not require the irrelevance of rejected contracts.

Lemma 4. *Suppose X blocks matching μ and choice functions satisfy substitutability. Then for every $x \in X \setminus \mu$, $\{x\}$ blocks μ.*

Proof. If X is a blocking set, then $X \subseteq C^s(\mu \cup X|\mu) \cap C^b(\mu \cup X|\mu)$. Take any $x \in X \setminus \mu$. Since choice function c_i satisfies substitutability, we have $r_i(\mu \cup \{x\}|\mu) \subseteq r_i(\mu \cup X|\mu)$ for every agent i. This implies $x \in c_i(\mu \cup \{x\}|\mu)$ for every i, so $x \in C^s(\mu \cup \{x\}|\mu) \cap C^b(\mu \cup \{x\}|\mu)$. Therefore, $\{x\}$ is a blocking set for μ. □

Appendix C: Proofs of Theorems 1 and 2

We start with the proofs of the two auxiliary lemmas from Section 4.2. We then first prove Theorem 2 (without using Theorem 1), and then use Theorem 2 to prove Theorem 1.

8.1 Proof of Lemma 2

Function f is monotonic in $\sqsubseteq$ because for any $A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^s \tilde{\mu}^s, \mu^b \succeq^b \tilde{\mu}^b$, substitutability implies that

$$\begin{aligned} \mathcal{X} \backslash R^b(A^b|\mu^b) &\subseteq \mathcal{X} \backslash R^b(\tilde{A}^b|\tilde{\mu}^b), \\ \mathcal{X} \backslash R^s(A^s|\mu^s) &\supseteq \mathcal{X} \backslash R^s(\tilde{A}^s|\tilde{\mu}^s), \end{aligned}$$

and consistency implies that

$$\begin{aligned} C^s(A^s|\mu^s) &\preceq^s C^s(\tilde{A}^s|\tilde{\mu}^s), \\ C^b(A^b|\mu^b) &\succeq^b C^b(\tilde{A}^b|\tilde{\mu}^b). \end{aligned}$$

Therefore, $(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$ implies that $f(A^s, A^b, \mu^s, \mu^b) \sqsubseteq f(\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$.

8.2 Proof of Lemma 3

$A^s \cup A^b = A^s \cup [\mathcal{X} \backslash R^s(A^s|\mu^s))] \supseteq A^s \cup [\mathcal{X} \setminus A^s] = \mathcal{X}$, so

$$A^s \cup A^b = \mathcal{X}.$$

Similarly, $A^s \cap A^b = A^s \cap [\mathcal{X} \backslash R^s(A^s|\mu^s))] = A^s \setminus R^s(A^s|\mu^s) = C^s\left(A^s|\mu^s\right)$, which implies $C^s\left(A^s|\mu^s\right) = A^s \cap A^b$. Analogously for b, $C^b\left(A^b|\mu^b\right) = A^s \cap A^b$. Finally, $\mu^\theta = C^\theta\left(A^\theta|\mu^\theta\right)$ implies

$$\mu^s = \mu^b = A^s \cap A^b = C^b(A^b|\mu^b) = C^s\left(A^s|\mu^s\right).$$

8.3 Proof of Theorem 2

First, suppose that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of function f. Claim 1 below shows that, under the hypothesis of Theorem 2, μ is a stable matching.

Claim 1. Suppose that choice functions satisfy substitutability and the irrelevance of rejected contracts. Then matching μ is stable.

Proof. Suppose for contradiction that μ is not stable. Then there are three possibilities, all of which we proceed to rule out.

1. Matching μ is not individually rational for some seller j, that is $c_j(\mu|\mu) \subsetneq \mu_j$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, $C^s(A^s|\mu) = \mu$ and $A^s \supseteq \mu$. But substitutability and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^s by agent j, that is $x \notin C^s\left(A^s|\mu\right)$, a contradiction.

2. Matching μ is not individually rational for some buyer i, that is $c_i(\mu|\mu) \subsetneq \mu_i$.This is analogous to the previous case since f treats buyers and sellers symmetrically.

3. There exists a blocking pair with contract $x \in X \setminus \mu$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3 $A^s \cup A^b = \mathcal{X}$. Therefore, without loss of generality, assume that $x \in A^b$. Since $\{x\}$ is a blocking set, there exists buyer i such that $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$. Again, since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3 $C^b(A^b|\mu) = \mu$, which implies that $c_i(A^b|\mu) = \mu_i$. By the irrelevance of rejected contracts, for any set Y such that $A^b \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular, for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}|\mu) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$.

To finish the proof of the theorem, we need to show that if matching μ is stable then there exist sets of contracts A^s, A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. The following is useful in our construction of A^s and A^b.

Claim 2. Suppose that choice functions satisfy substitutability and the irrelevance of rejected contracts. Then the function $M^\theta\left(\mu\right) \equiv \max\{X \subseteq \mathcal{X}|C^\theta\left(X|\mu\right) = \mu\}$, where the maximum is with respect to set inclusion, is well defined. Moreover, for any contract $x \in M^\theta\left(\mu\right)$, $x \in C^\theta(M^\theta\left(\mu\right) \cup x|\mu)$.

Proof. If there are two sets M' and M'' such that $C^{\theta}(M'|\mu) = C^{\theta}(M''|\mu) = \mu$, then (by substitutability)

$$\begin{aligned} C^{\theta}(M' \cup M''|\mu) &= (M' \cup M'') \setminus R^{\theta}(M' \cup M''|\mu) = \left[M' \setminus R^{\theta}(M' \cup M''|\mu)\right] \cup \left[M'' \setminus R^{\theta}(M' \cup M''|\mu)\right] \\ &\subseteq \left[M' \setminus R^{\theta}(M'|\mu)\right] \cup \left[M'' \setminus R^{\theta}(M''|\mu)\right] = \mu. \end{aligned}$$

If $C^{\theta}(M' \cup M''|\mu)$ was a proper subset of μ, then the irrelevance of rejected contracts would imply that $C^{\theta}(M'|\mu) = C^{\theta}(M''|\mu) = C^{\theta}(M' \cup M''|\mu)$, which is a contradiction. Therefore, $M^{\theta}(\mu)$ is well defined. Let $x \notin M = M^{\theta}(\mu)$. If $x \notin C^{\theta}(M \cup x|\mu)$, then $C^{\theta}(M \cup x|\mu) = C^{\theta}(M|\mu)$ by the irrelevance of rejected contracts. But this implies $C^{\theta}(M \cup x|\mu) = \mu$, which contradicts maximality of M. Hence $x \in C^{\theta}(M \cup x|\mu)$.

Claim 3. Suppose that the matching μ is stable and the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then there exist sets of contracts A^s and A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f.

Proof. By Claim 2, there exists the largest set $M^{\theta}(\mu) \equiv \max\{X \subseteq \mathcal{X} | C^{\theta}(X|\mu) = \mu\}$. Let $A^s \equiv M^s(\mu)$ and $A^b \equiv \mathcal{X} \setminus R^s(A^s|\mu)$. By definition, $A^b = \mathcal{X} \setminus R^s(A^s|\mu)$ and $\mu = C^s(A^s|\mu)$. Thus, we get $A^s \cap A^b = A^s \cap (\mathcal{X} \setminus R^s(A^s|\mu)) = C^s(A^s|\mu) = \mu$. To finish the proof, we need to show $\mu = C^b(A^b|\mu)$ and $A^s = \mathcal{X} \setminus R^b(A^b|\mu)$.

Note that $A^b = \mathcal{X} \setminus R^s(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup C^s(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup \mu$. In particular, $A^b \supseteq \mu$. If $C^b(A^b|\mu) = Y \neq \mu$, there are two cases, both of which contradict stability of μ. First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^b(\mu|\mu) = Y$, implying that μ is not individually rational for some buyers, contradicting stability. Second, if $Y \nsubseteq \mu$, then there exists a $y \in Y \setminus \mu$, and $y \in C^b(\mu \cup \{y\}|\mu)$ by substitutability since $y \in C^b(A^b|\mu)$ and $A^b \supseteq \mu \cup \{y\}$. But we also have that $y \in C^s(A^s \cup \{y\}|\mu)$ by Claim 2. Then $\{y\}$ blocks μ, contradicting stability. Thus, the only case consistent with stability is $C^b(A^b|\mu) = \mu$.

Finally, we show that $A^s = \mathcal{X} \setminus R^b(A^b|\mu) = \mathcal{X} \setminus R^b(\mathcal{X} \setminus R^s(A^s|\mu)|\mu)$. Since $C^b(A^b|\mu) = \mu$, then $\mathcal{X} \setminus R^b(A^b|\mu) = \mathcal{X} \setminus (A^b \setminus \mu) = \mathcal{X} \setminus (((\mathcal{X} \setminus A^s) \cup \mu) \setminus \mu) = \mathcal{X} \setminus (\mathcal{X} \setminus A^s) = A^s$ and we have the result.

8.4 Proof of Theorem 1

First, let us consider the first phase of the algorithm and check that $\mu^* \geq^s C^s(\mathcal{X}|\mu^*)$. By the irrelevance of rejected contracts, we get $C^s(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \geq^s \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathcal{X} \supseteq \emptyset$ and consistency imply that

$$\mu_1 = C^s(\mathcal{X}|\emptyset) \geq^s C^s(\emptyset|\emptyset) = \emptyset = \mu_0.$$

For the general case, $\mu_k \geq^s \mu_{k-1}$ and $\mathcal{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^s(\mathcal{X}|\mu_k) \geq^s C^s(\mu_k|\mu_{k-1}) = \mu_k.$$

Therefore, $\{\mu_k\}_{k\geq 1}$ is a monotone sequence with respect to the preorder $\geq^s$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$; we take the minimum m satisfying this property and set $\mu^* = \mu_m$. Then,

$$C^s(\mathcal{X}|\mu_m) = \mu_{m+1} = \mu_n \leq^s \mu_m$$

where the monotonicity comparison follows as $\leq^s$ is transitive.

It remains to show that the second phase converges and that the resulting matching is stable. It is easy to see that $f(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsubseteq (\mathcal{X}, \emptyset, \mu^*, \emptyset)$, since $C^s(\mathcal{X}|\mu^*) \leq^s \mu^*$ by construction and $C^b(\emptyset|\emptyset) = \emptyset \geq^b \emptyset$ by reflexivity of $\geq^b$. By Lemma 2, f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^k(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsubseteq f^{k-1}(\mathcal{X}, \emptyset, \mu^*, \emptyset)$ for every k. We consider two separate cases. Suppose first that this sequence converges. Therefore, there exists k such that $f^{k-1}(\mathcal{X}, \emptyset, \mu^*, \emptyset) = f^k(\mathcal{X}, \emptyset, \mu^*, \emptyset)$. As a result, $f^{k-1}(\mathcal{X}, \emptyset, \mu^*, \emptyset)$ is a fixed point of f. Let $(A^{*s}, A^{*b}, \mu^{*s}, \mu^{*b}) \equiv f^{k-1}(\mathcal{X}, \emptyset, \mu^*, \emptyset)$. By Lemma 3, $\mu^{*s} = \mu^{*b} = A^{*s} \cap A^{*b}$ and μ^{*b} is a stable matching by Theorem 2.

Otherwise, if the sequence does not converge, there exists a subsequence $f^n(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^{n+1}(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq \ldots \sqsupseteq f^m(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^{m+1}(\mathcal{X}, \emptyset, \mu^*, \emptyset) = f^n(\mathcal{X}, \emptyset, \mu^*, \emptyset)$ because the number of contracts is finite. By transitivity of the preorder $\sqsupseteq$ and the previous inequality, we get $f^n(\mathcal{X}, \emptyset, \mu^*, \emptyset) = f^{m+1}(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^m(\mathcal{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^n(\mathcal{X}, \emptyset, \mu^*, \emptyset)$. Let $f^n(\mathcal{X}, \emptyset, \mu^*, \emptyset) = (A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ and $f^m(\mathcal{X}, \emptyset, \mu^*, \emptyset) = (A_2^s, A_2^b, \mu_2^s, \mu_2^b)$. By definition of $\sqsupseteq$, we get that $A_1^s = A_2^s$, $A_1^b = A_2^b$, $\mu_1^s \sim^s \mu_2^s$, and $\mu_1^b \sim^b \mu_2^b$. Now, by construction $C^s(A_2^s|\mu_2^s) = \mu_1^s$ and by substitutability $C^s(A_2^s|\mu_2^s) = C^s(A_1^s|\mu_1^s)$, which imply that $C^s(A_1^s|\mu_1^s) = \mu_1^s$. Similarly, we get that $C^s(A_1^s|\mu_1^b) = \mu_1^b$. Furthermore, by substitutability, $\mathcal{X} \backslash R^b(A_2^b|\mu_2^b) = \mathcal{X} \backslash R^b(A_1^b|\mu_1^b)$ and, by construction, $\mathcal{X} \backslash R^b(A_2^b|\mu_2^b) = A_1^b$, which imply $\mathcal{X} \backslash R^b(A_1^b|\mu_1^b) = A_1^b$. Similarly, we get $\mathcal{X} \backslash R^s(A_1^s|\mu_1^s) = A_1^s$. Therefore, $(A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ is a fixed point of f. This shows that the sequence converges as in the previous paragraph, so there exists a stable matching.