Pay-as-Bid: Selling Divisible Goods

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Abstract

Pay-as-bid is the most popular auction format for selling treasury securities. We prove the uniqueness of pure-strategy Bayesian-Nash equilibria in pay-as-bid auctions where symmetrically-informed bidders face uncertain supply, and we establish a tight sufficient condition for the existence of this equilibrium. Equilibrium bids have a convenient separable representation: the bid for any unit is a weighted average of marginal values for larger quantities. With optimal supply and reserve price, the pay-as-bid auction is revenue-equivalent to the uniform-price auction.

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1 Introduction

The pay-as-bid auction, also known as the discriminatory auction, is among the most commonly used auction formats: each year, securities and commodities worth trillions of dollars are traded in pay-as-bid auctions. It is the most popular auction format for selling treasury securities, with 33 out of 48 countries in a recent survey using pay-as-bid auctions to sell their securities.[1] The pay-as-bid format is also used in other government operations, including the recent large-scale asset purchases in the U.S. (known as Quantitative Easing), and it is frequently used to allocate commodities such as electricity.[2]

Our paper provides a general theory of equilibrium bidding in pay-as-bid auctions: we provide a tight sufficient condition for the existence of Bayesian-Nash equilibrium, we prove that this equilibrium is unique, and we offer a surprising closed-form bid representation theorem. In our model, bidders are symmetrically-informed and uncertain of the total supply available for auction.[3] Uncertainty over supply is a feature of many securities auctions, while symmetric information is a simplifying assumption which seems to be a good approximation in some important environments. For instance, any issue of treasury securities has both close substitutes whose prices are known, and the forward contracts based on the issue are traded ahead of the auction in the forward markets, thus providing bidders with substantial information about each others' valuations.[4]

We leverage our theory of equilibrium bidding to show how to optimally design supply and reserve prices in pay-as-bid auctions, and to prove that pay-as-bid auctions and uniform-price auctions (the main alternative auction format) are revenue equivalent when supply and reserve prices are designed optimally. Our revenue equivalence result might explain why the long-standing debate over which of these two auction formats is revenue superior has not been settled thus far. In our discussion of auction design, we allow the seller to not know the bidders' values; we thus allow situations in which bidders'

[1]See Brenner et al. [2009].

[2]Quantitative Easing used a reverse pay-as-bid auction. For a recent discussion of electricity markets see Maurer and Barroso [2011]. A pay-as-bid auction is implicitly run in financial markets when limit orders are followed by a market order (see e.g. Glosten [1994]).

[3]We also assume that the traded good is perfectly divisible, a good approximation of treasury and commodity auctions in which thousands, or even millions, of identical units are sold simultaneously. We allow an arbitrary finite number of bidders.

[4]Conversely, our assumptions do not present a good approximation in other environments, in which there may be substantive asymmetries among bidders. Nonetheless, our assumptions are milder than those typically employed by the rich prior literature on pay-as-bid auctions (see below for a discussion): with the exception of flat-demand environments, single-unit demand, and two-bidder-two-units examples, all known examples of equilibrium bidding restrict attention to symmetrically-informed bidders.

information was not available to the seller at the time the auction was designed, as well as situations in which the seller designs the auction format once and uses it for many auctions.

Prior work, most notably Wang and Zender [2002], Holmberg [2009], and Ausubel et al. [2014], proved equilibrium existence under substantially more restrictive assumptions than ours, and analyzed equilibria assuming linear marginal values and Pareto distribution of supply; in contrast we provide a general analysis with no parametric assumptions. Our bid representation theorem is surprising in the context of this prior literature, a natural reading of which is that equilibria in pay-as-bid auctions with symmetrically informed bidders are complex. We discuss this literature below in more detail. Our work on the optimal design of supply and reserve prices has no direct counterpart in the prior literature.[5]

Before describing our results and the rich related literature in more detail, we describe how the pay-as-bid auction is run. First, the bidders submit bids for each infinitesimal unit of the good. Then, the supply is realized, and the auctioneer (or, the seller) allocates the first infinitesimal unit to the bidder who submitted the highest bid, then the second infinitesimal unit to the bidder who submitted the second-highest bid, etc.[6] Each bidder pays her bid for each unit she obtains. The monotonic nature of how units are allocated implies that we can equivalently describe a collection of bids a bidder submitted as a reported demand curve that is weakly-decreasing in quantity, but not necessarily continuous; the ultimate allocation resembles that of a classical Walrasian market, in which supply equals demand at a market-clearing price. We study pure-strategy Bayesian-Nash equilibria of this auction.[7]

The theory of equilibrium bidding we develop has three components. First, we establish a sufficient condition for equilibrium existence. The sufficient condition is expressed in terms of primitives of the model and is relatively simple to check. The condition is

[5] In particular, even in previously-studied parametric models our analysis of optimal supply and reserve price leads beyond existing results.

[6] To fully-specify the auction we need to specify a tie-breaking rule; we adopt the standard tie-breaking rule, pro-rata on the margin, but our theory of equilibrium bidding does not hinge on this choice. This is in contrast to uniform-price auction, where tie-breaking matters; see Kremer and Nyborg [2004].

[7] In equilibrium, each bidder responds to the stochastic residual supply (that is, the supply given the bids of the remaining bidders). Effectively, the bidder is picking a point on each residual supply curve. In determining her best response, a bidder needs to keep in mind that: (i) the bid that is marginal if a particular residual supply curve is realized is paid not only when it is marginal, but also in any other state of nature that results in a larger allocation, and hence the bidder faces tradeoffs across these different states of nature; and (ii) the bids need to be weakly monotonic in quantity, potentially a binding constraint.

generically tight in that its weak-inequality analogue is necessary for equilibrium existence. Our condition is satisfied, for instance, in the linear-Pareto settings analyzed by the prior literature, and it is satisfied for linear marginal values and any distribution of supply provided there are sufficiently many bidders.[8]

Second, we prove that there is a unique pure-strategy Bayesian-Nash equilibrium in pay-as-bid auctions, conditional on its existence.[9] The uniqueness of equilibrium is reassuring for sellers using the pay-as-bid format; indeed, there are well-known problems posed by multiplicity of equilibria in other multi-unit auctions.[10] Uniqueness is also important for the empirical study of pay-as-bid auctions. Estimation strategies based on the first-order conditions, or the Euler equation, rely on agents playing comparable equilibria across auctions in the data (Février et al. [2002], Hortaçsu and McAdams [2010], Hortaçsu and Kastl [2012], and Cassola et al. [2013]).[11] Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see Armantier and Sbaï [2006]).[12] The uniqueness of equilibrium provides a theoretical foundation for these estimation strategies and counterfactual analysis.

Our third result regarding equilibrium bidding is the bid representation theorem. We show that in the unique pure-strategy Bayesian-Nash equilibrium the bid for any quantity is a weighted average of the bidder's marginal values for this and larger quantities, where the weights are independent of the bidder's marginal values.[13] The weighting distribution depends only on the distribution of supply and the number of bidders. The tail of the weighting distribution is equal to the tail of the distribution of supply scaled by a factor

[8]For many distributions of interest our condition is also satisfied with relatively few bidders; see our examples throughout the paper.

[9]We establish the uniqueness of bids for relevant quantities—that is, for quantities a bidder wins with positive probability. Bids for higher quantities play no role in equilibrium as long as they are not too low. Our uniqueness result and the subsequent discussion does not apply to these irrelevant bids. Also, since we work in a model with continuous quantities we do not distinguish between two bid functions that coincide almost everywhere; we can alter an equilibrium bid function on a measure-zero subset of quantities without affecting equilibrium outcomes.

[10]The uniform-price auction—the other frequently used auction format—can admit multiple equilibria, some of which generate very little revenue. See LiCalzi and Pavan [2005], McAdams [2007], Kremer and Nyborg [2004], and the discussion of uniform-price below. There is no contradiction here with our revenue equivalence presented below: we prove revenue equivalence between the unique equilibrium in pay-as-bid and the seller-optimal equilibrium in uniform-price; the seller can ensure that the latter equilibrium is unique by judiciously selecting the reserve price.

[11]Maximum likelihood-based estimation strategies (e.g. Donald and Paarsch [1992]) also rely on agents playing comparable equilibria across auctions in the data. Chapman et al. [2005] discuss the requirement of comparability of data across auctions.

[12]See also, in a related context, Cantillon and Pesendorfer [2006]

[13]Swinkels [2001] showed a large-market limit counterpart of this insight in a different model. In contrast, our results obtain exactly in all finite markets and not only in the limit.

that depends only on the number of bidders; an increase in the number of bidders shifts the weight away from the tail and towards the unit for which the bid is submitted, hence increasing the bid for this unit.

The bid representation theorem implies several properties of equilibrium bidding: the unique equilibrium is symmetric and the bid functions are strictly decreasing and differentiable in quantity. In our analysis we allow all Bayesian-Nash equilibria, including asymmetric ones, and we impose no strict-monotonicity or regularity assumptions on the submitted bids; we thus prove symmetry, strict monotonicity, and differentiability rather than assuming them. Furthermore, with the distribution of supply concentrated around a target quantity, our representation implies that bids are nearly flat for units lower than the target, and that the bidder's margin on units near the target is low. The representation theorem also implies that the seller's revenue increases when bidders' values increase, or when more bidders arrive. Finally, the bid representation theorem plays a central role in our analysis of auction design and revenue equivalence between pay-as-bid and uniform-price auctions.

Building on our theory of equilibrium bidding, we address outstanding questions surrounding the design of divisible-good auctions. Traditionally a key instrument in auction design is the reserve price; for divisible goods there is a second natural instrument: the supply distribution. In the special case when all information is publicly available, every reserve price decision can be replicated by an appropriate supply restriction, so that the two choices lead to identical bidding behavior in the unique Bayesian-Nash equilibrium of the pay-as-bid auction.[14] When bidders have information that is not available to the seller, however, both the supply restriction and the reserve price play important role in revenue maximization.[15]

Our main result on revenue-maximizing supply distributions in pay-as-bid auctions says that, regardless of the information structure, the revenue in the unique pure-strategy equilibrium is maximized when supply is deterministic; computing the level of the optimal deterministic supply is equivalent to a standard monopoly problem.[16] In practice, in

[14]We show that supply adjustments can accomplish any design objective that can be achieved with reserve prices, but—as we also show—the reverse is not true. In this regard pay-as-bid is different from uniform-price; as we discuss below, reserve prices play an important role in uniform-price auctions.

[15]This is due to a natural complementarity between the two features: given a reserve price, optimizing supply will generally improve revenue; given a distribution of supply, optimizing the reserve price will generally improve revenue. Generally, either of the two instruments may be more valuable.

[16]Because the seller in our model can set both a limiting quantity and limiting price, this monopoly problem is not entirely "standard." Nonetheless, it is straightforward to envision a monopolist setting

many of these auctions the distribution of supply is partially determined by the demand from non-competitive bidders, and revenue maximization is not the only objective of the sellers. However, treasuries and central banks have the ability to influence the supply distributions, as well as to release data on non-competitive bids to competitive bidders; in this context our result provides a revenue-maximizing benchmark.

While the result that deterministic selling strategies are optimal is familiar from the no-haggling theorem of Riley and Zeckhauser [1983], in multi-object settings the reverse has been shown by Pycia [2006]. Furthermore, there is a subtlety specific to pay-as-bid that might suggest a role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to bid more on initial units, and in pay-as-bid the seller collects the raised bids even when the realized supply is near monopoly quantity. We show that, despite these considerations, committing to deterministic supply is indeed optimal.

Our last major result compares the revenues generated by pay-as-bid and uniform-price auctions. If the the seller knows the bidders' values then our result on the optimality of deterministic supply allows us to easily show that with supply and reserve prices chosen optimally in both auction formats, the two formats are revenue-equivalent.[17] We further show that this revenue equivalence holds true regardless of what the seller knows if we restrict attention to seller's optimal equilibria in the uniform-price auction. For other uniform-price auction equilibria, we show that the pay-as-bid auction is unambiguously revenue dominant. This provides novel evidence in favor of pay-as-bid auctions. When the distribution of supply and the reserve price are near their optima equilibrium revenue will be nearly optimal. Since equilibrium in the pay-as-bid auction is unique regardless of the reserve price while the uniform-price auction admits collusive-seeming equilibria with revenues equal to the (nonoptimal) reserve, the pay-as-bid auction eliminates the potential selection of seller-pessimal equilibria, while ensuring approximately optimal seller revenues.[18]

the limiting price and the limiting quantity.

[17]Setting the reserve price does not affect equilibrium selection in the pay-as-bid auction, but is important in the uniform-price auction to ensure that a seller-preferred equilibrium arises. With supply and reserve price set optimally, the uniform-price auction has a unique equilibrium; were we to only set the supply optimally and ignore the reserve price, the uniform-price auction could have multiple equilibria and the revenue equivalence with the unique equilibrium of pay-as-bid would obtain only for the revenue-maximizing equilibrium of uniform-price.

[18]A related point is made by Hortaçsu et al. [2016], who note that bids in U.S. Treasury auctions are typically "flat" and infer that not much surplus is retained by bidders. In our model this implies that there is not much difference between the revenues generated by the pay-as-bid and uniform-price

Paper	Data	Method	Conclusion	σ/μ
FPV (2002)	France	CF PABA $\to$ UPA	PABA $>$ UPA	1.27%
AS (2006)	France	CF PABA $\to$ UPA	UPA $>$ PABA	3.78%
Umlauf (1993)	Mexico	Natural experiment	UPA $>$ PABA	11.16%

Table 1: Revenue comparisons between auction formats, in comparison to the standard deviation of noncompetitive demand scaled by mean aggregate supply (Q); "CF" is "counterfactual."

Our divisible-good revenue equivalence result provides a benchmark for the long-standing debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial attention in empirical structural IO, with Hortaçsu and McAdams [2010] finding no statistically significant differences in revenues, Février et al. [2002] and Kang and Puller [2008] finding slightly higher revenues in pay-as-bid, and Castellanos and Oviedo [2004], Armantier and Sbaï [2006], and Armantier and Sbaï [2009] finding slightly higher revenues in uniform-price. Our revenue equivalence result provides a possible explanation for this surprising pattern;[19] Table 1 relates the revenue comparisons in the literature to normalized randomness in aggregate supply, and suggests that increased uncertainty in supply improves the relative performance of UPA over PABA.[20]

Prior theoretical work on the pay-as-bid versus uniform-price question has focused on revenue comparisons for fixed supply distributions, has allowed for neither reserve price nor supply optimization, and has assumed that the seller is perfectly informed about buyers' values. Wang and Zender [2002] find pay-as-bid revenue superior in the equilibria of the linear-Pareto model their consider. Ausubel et al. [2014] show that—with asymmetric bidders—either format can be revenue superior; with symmetric bidders pay-as-bid is revenue superior in all examples they consider. The special supply distributions

auctions, but the uniform-price auction brings large downside potential in the form of collusive-seeming behavior. Implementing a pay-as-bid auction mitigates this risk.

[19]These papers tend to take a counterfactual approach to comparing auction formats: data is generated in pay-as-bid auctions; bidder values are inferred using first-order conditions; counterfactual equilibrium bids are then derived for the uniform-price auction; and then revenue is compared. Our revenue equivalence provides an explanation for the near-revenue equivalence suggested by the data even if in the counterfactual part of the analysis the reserve price and supply are not optimized because the optimal reserve price and supply are the same in both pay-as-bid and uniform-price formats. Furthermore, the insight obtains even if no reserve price is imposed in the uniform-price counterfactual because in pay-as-bid we only observe bids which are above the reserve price.

[20]The small number of results summarized in Table 1 compared to the larger number of results in the previous paragraph is a matter of data availability. The papers in Table 1 separately summarize aggregate and noncompetitive supply and provide the first two moments, allowing a simple calculation of the relative randomness of a single run of an auction.

these papers consider are not revenue-maximizing, hence there is no conflict between their strict rankings and our revenue equivalence. Swinkels [2001] showed that pay-as-bid and uniform-price are revenue-equivalent in large markets; our equivalence result does not rely on the size of the market.

Finally, let us note that our analysis of pay-as-bid auctions can be reinterpreted as a model of dynamic oligopolistic competition among sellers who at each moment of time compete a la Bertrand for sales and who are uncertain how many more buyers are yet to arrive. Prior sales determine the production costs for subsequent sales, thus the sellers need to balance current profits with the change in production costs in the future. This methodological link between pay-as-bid auctions and dynamic oligopolistic competition is new, and we develop it in our follow up work.[21]

1.1 Literature

There is a large literature on equilibrium existence in pay-as-bid auctions. The existence of pure-strategy equilibria has been demonstrated when the marginal values are linear and the distribution of supply is Pareto, see Wang and Zender [2002], Federico and Rahman [2003], and Ausubel et al. [2014]. Holmberg [2009] proved the existence of equilibrium when the distribution of supply has a decreasing hazard rate and he recognized the possibility that equilibrium may not exist.[22] Our sufficient condition for existence encompasses the prior conditions and is substantially milder; in fact, with a sufficient number of bidders all distributions, including for instance the truncated normal distribution, satisfy our condition.[23]

Equilibrium existence has also been proved in settings with private information. Woodward [2016] proved the existence of pure-strategy equilibria in pay-as-bid auctions of perfectly-divisible goods. Earlier work establishing the existence of pure-strategy equilibria has looked at multi-unit (discrete) settings—see, for instance, Athey [2001], McAdams [2003], and Reny [2011].[24] A key difference between these papers and ours is

[21]The oligopolistic sellers uncertain of future demands correspond to bidders in the pay-as-bid auction, and sellers' costs correspond to bidders' values. For prior studies of dynamic competition see e.g. **?**; while they study competition among a continuum of sellers, the pay-as-bid-based approach allows for the strategic interaction between a finite number of sellers. The other canonical multi-unit auction format, the uniform-price auction, was earlier interpreted in terms of static oligopolistic competition by **?**.

[22]See also Fabra et al. [2006], Genc [2009], and Anderson et al. [2013] for discussions of potential problems with equilibrium existence.

[23]Prior literature conjectured that equilibrium cannot exist for truncated normal distributions.

[24]See also Břeský [1999], Jackson et al. [2002], Reny and Zamir [2004], Jackson and Swinkels [2005],

that the presence of private information allows the purification of mixed-strategy equilibria; such purification is not possible in our setting.

Uniqueness was studied by Wang and Zender [2002] who proved the uniqueness of "nice" equilibria under strong parametric assumptions on utilities and distributions. Assuming that marginal values are linear and the supply is drawn from an unbounded Pareto distribution, they analyzed symmetric equilibria in which bids are piecewise continuously-differentiable functions of quantities and supply is invertible from equilibrium prices; they showed the uniqueness of such equilibria. Holmberg [2009] restricted attention to symmetric equilibria in which bid functions are twice differentiable, and—assuming that the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy—proved the uniqueness of such smooth and symmetric equilibria.[25] Ausubel et al. [2014] expanded the previous analysis to Pareto supply with bounded support and linear marginal values. Restricting attention to equilibria in which bids are linear functions of quantities, they showed the uniqueness of such linear equilibria. In contrast, we look at all Bayesian-Nash equilibria of our model, we impose no parametric assumptions and we do not require that some part of the supply is not wanted by any bidder.[26]

Our uniqueness result is also related to Klemperer and Meyer [1989] who established uniqueness in a duopoly model closely related to uniform-price auctions: when two symmetric and uninformed firms face random demand with unbounded support, then there is a unique equilibrium in their model.[27] The main difference between the two papers is, of course, that Klemperer and Meyer analyze the uniform-price auction, while we look

Břeský [2008], and Kastl [2012]. Milgrom and Weber [1985] showed existence of mixed-strategy equilibria.

[25]Holmberg's assumption that bidders do not want to buy part of the supply represents a physical constraint in the reverse pay-as-bid electricity auction he studies: in his paper the bidders supply electricity and they face a capacity constraint—beyond a certain level they cannot produce more. This low-capacity assumption drives the analysis and it precludes directly applying the same model in the context of securities auctions in which the bidders are always willing to buy more provided the price is sufficiently low.

[26]As a consequence of this generality, we need to develop a methodological approach which differs from that of the prior literature. McAdams [2002] and Ausubel et al. [2014] have also established the uniqueness of equilibrium in their respective parametric examples with two bidders and two goods.

[27]The analogue of their unbounded support assumption is our assumption that the support of supply extends all the way to no supply. While the two assumptions look analogous they have very different practical implications: a seller can guarantee that with some tiny probability the supply will be lower than the target; in fact, in practice the supply is often random and our support assumption is satisfied. On the other hand, it is substantially more difficult, and practically impossible, for the seller to guarantee the risk of arbitrarily-large supplies. Note also that we have known since Wilson [1979] that the uniform-price auction may admit multiple equilibria. No similar multiplicity constructions exist for pay-as-bid auctions.

at pay-as-bid.[28]

Prior constructions of equilibria focused on the setting in which bidders' marginal values are linear in quantity and the distribution of supply is (a special case of) the generalized Pareto distribution; see Wang and Zender [2002], Federico and Rahman [2003], and Ausubel et al. [2014]. This literature expressed equilibrium bids in terms of the intercept and slope of the linear demand and the parameters of the generalized Pareto distribution. In addition to studying the linear-Pareto setting, Holmberg [2009] formulated a general first-order condition satisfied by symmetric smooth equilibria, and he solved it under the assumption that the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy.[29] We do not rely on any of these assumptions, and our representation of bids as weighted averages of marginal values is new.

While we are not aware of prior literature on optimal design of supply and reserve prices in pay as bid, a more general question was addressed by Maskin and Riley [1989]: what is the revenue-maximizing mechanism to sell divisible goods? The optimal mechanism they described is complex and in practice the choice seems to be between the much simpler auction mechanisms: pay-as-bid and uniform-price.[30] Above, we discussed the literature on revenue comparisons between these two popular mechanisms.

2 Model

There are $n \geq 2$ bidders, $i \in \{1, ..., n\}$. Each bidder's marginal valuation for quantity q is denoted $v^i(q; s) = v(q; s)$, where s is a signal commonly known to all bidders but not to the seller. We assume that v is strictly decreasing, Lipschitz continuous, and almost-everywhere differentiable in q. Because the signal s is commonly known by all bidders it is not of strategic importance: when studying the equilibrium among bidders in Sections 3 and 4, we therefore fix s and denote the bidders' marginal valuation by $v^i(q) = v(q)$; bidders' information will be important in the analysis of the seller's problem in Sections

[28] Our uniqueness result is also related to papers on uniqueness of equilibria in single-unit auctions, particularly in first-price auctions; see, for instance, Maskin and Riley [2003], Lizzeri and Persico [2000], and Lebrun [2006].

[29] See footnote 25. We focus our discussion on settings with decreasing marginal utilities; for constant marginal utilities see Back and Zender [1993] and Ausubel et al. [2014] among others.

[30] Design issues were addressed in the context of uniform price auction: Kremer and Nyborg [2004] looks at the role of tie-breaking rules, LiCalzi and Pavan [2005] at elastic supply, and McAdams [2007] at commitment.

5 and 6.[31] We allow arbitrary distribution of s, and an arbitrary integrable $v(q;\cdot)$.

The supply Q is drawn from a non-degenerate distribution F with density f and support $[0,\overline{Q}]$; we assume that Q is independent of the bidders' signal s.[32] We assume that $f>0$ on the support and otherwise we impose no global assumptions on F. In particular, we allow distributions that are concentrated around some quantity and take values close to 0 with arbitrarily small probability.[33] We denote the inverse hazard rate by $H=\frac{1-F}{f}$.

In the pay-as-bid auction, each bidder submits a weakly decreasing bid function $b^i(q):[0,\overline{Q}]\to\mathbb{R}_+$. Without loss of generality we may assume that the bid functions are right-continuous.[34] The auctioneer then sets the market price p (also known as the stop-out price),

$$p=\sup\left\{p':\ q^1+...+q^n\geq Q \text{ for all } q^1,\ldots,q^n \text{ such that } \ b^1\left(q^1\right),...,b^n\left(q^n\right)\leq p'\right\}.$$

Agents are awarded a quantity associated with their demand at the stop-out price,

$$q^i=\max\left\{q':\ b^i\left(q'\right)\geq p\right\},$$

as long as there is no need to ration them. When necessary, we ration pro-rata on the margin, the standard tie-breaking in divisible-good auctions. The details of the rationing rule have no impact on the analysis of equilibrium bidding we pursue in Section 3.[35] The demand function (the mapping from p to q^i) is denoted by $\varphi^i(\cdot)$. Agents pay their bid

[31] The seller may not know the bidders' information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

[32] For instance, Q might represent supply net of non-competitive bids as discussed in Back and Zender [1993], Wang and Zender [2002], and subsequent literature. Our uniqueness result does not rely on any additional assumptions, while our equilibrium existence theorem specifies a sufficient condition for the existence.

[33] In some results, we also consider the limit as F puts all mass on a single quantity.

[34] This assumption is without loss because we study a perfectly-divisible good and we ration quantities pro-rata on the margin. Indeed, we could alternatively consider an equilibrium in strategies that are not necessarily right-continuous. By assumption, the equilibrium bid function of a bidder is weakly decreasing, hence by changing it on measure zero of quantities we can assure the bid function is right continuous. Such a change has no impact on this bidder's profit, or on the profits of any of the other bidders, because rationing pro-rata on the margin is monotonic in the sense of footnote 35. In fact, there is no impact on bidders' profits even conditional on any realization of Q.

[35] The only place when we rely on rationing rule is the analysis of reserve prices we present in Section 5.1. Even in this Section all we need is that rationing rule is monotonic: that is, the quantity assigned to each bidder increases when the stop-out price decreases; rationing pro-rata on the margin satisfies this property.

for each unit received, and utility is quasilinear in monetary transfers; hence,

$$u^i\left(b^i\right) = \int_0^{q^i(p)} v\left(x\right) - b^i\left(x\right) dx.$$

We study Bayesian-Nash equilibria in pure strategies.

3 Existence, Uniqueness, and The Bid Representation Theorem

Let us start by introducing the central notion of weighting distributions. For any quantity $Q \in [0, \overline{Q})$, the n-bidder weighting distribution of F has c.d.f. $F^{Q,n}$ that increases from 0 when $x = Q$ to 1 when $x = \overline{Q}$. This c.d.f. is given by

$$F^{Q,n}\left(x\right) = 1 - \left(\frac{1 - F\left(x\right)}{1 - F\left(Q\right)}\right)^{\frac{n-1}{n}}.$$

The auxiliary c.d.f.s $F^{Q,n}$ play a central role in our bid representation theorem (see below) and throughout our paper. Importantly, these distributions depend only the number of bidders and the distribution of supply, and not on any bidder's true demand. Note that as the number of bidders increases the weighting distributions put more weight on lower quantities.

Given an inverse bid function φ, let $Y(q;b)$ be given by

$$Y\left(q;b\right) = \frac{1 - F\left(q + \left(n-1\right)\varphi\left(b\right)\right)}{f\left(q + \left(n-1\right)\varphi\left(b\right)\right)}.$$

Y is the inverse hazard rate H evaluated at the total quantity demanded at a price of b if one agent demands q units and all others submit the (inverse) bid function φ. We establish separately in Theorem 3 that equilibrium must be symmetric, so the assumption of symmetric behavior of a bidder's opponents is sufficient to analyze equilibrium existence. Our condition for existence is given in Theorem 1.

Theorem 1. [Existence] *Let φ be the inverse bid function corresponding to the bid function in equation 1. There exists a pure-strategy Bayesian-Nash equilibrium whenever,* *for all* $Q < \overline{Q}$ *and all* $p \in (p(0), p(\overline{Q}))$,

$$E\left(\frac{1}{n}Q\right) = (n-1)\left(v\left(\frac{1}{n}Q\right) - p\right)\varphi_p(p) + Y\left(\frac{1}{n}Q; p\right) = 0$$
$$\implies E_q\left(\frac{1}{n}Q\right) = \frac{v_q\left(\frac{1}{n}Q\right)Y(\varphi(p); p)}{p - v(\varphi(p))} + Y_q\left(\frac{1}{n}Q; p\right) > 0.$$

We provide the proof of Theorem 1 and subsequent results in the Appendix. The function E represents the equilibrium (negative) first-order conditions in the pay-as-bid auction; $E_q$is the cross-partial derivative of bidder utility with respect to bid and quantity.[36] Since $Y \geq 0$ everwhere the implication in Theorem 1 is equivalent to

$$Y(\varphi(p); p)\, v_q\left(\frac{1}{n}Q\right) - Y_q\left(\frac{1}{n}Q; p\right)\left(v\left(\frac{1}{n}Q\right) - p\right) < 0.$$

This resembles a standard second-order condition: the marginal gains to increasing the quantity bid-for at a particular price are strictly decreasing.

Although the existence condition in Theorem 1 depends on equilibrium (inverse) bids, it is fully determined by the primitives of the pay-as-bid auction. Theorem 3 gives an explicit form for unique equilibrium bids, conditional on existence, therefore the $Y(\varphi(p); p)$ term can be explicitly computed from marginal values v and the distribution of aggregate supply F.

Consider some examples. Our sufficient condition is satisfied when marginal values v are linear and F is a uniform distribution or a generalized Pareto distribution, $F(x) = 1 - \left(1 - \frac{x}{\overline{Q}}\right)^{\alpha}$ where $\alpha > 0$.[37] With linear marginal values, this condition is also satisfied for any distribution F provided there are sufficiently many bidders. Indeed, with linear marginal values the left-hand term is negative and constant (fixing $\varphi(p)$) while the right-hand term is decreasing in n.[38] And, the sufficient condition is satisfied whenever the inverse hazard rate H is increasing—hence when the hazard rate is decreas-

[36] The cross-partial derivative in this context fills the role of a second derivative in a classical context. If whenever the first-order condition is satisfied—whenever $E(Q/n) = 0$—the derivative of the first-order condition with respect to its parameter (q) is strictly negative, there can be only one q at which the first-order condition is satisfied for any b. Then there is at most one b at which the first-order condition is satisfied for any q.

[37] The existence of equilibrium in the linear/generalized Pareto example was established by Ausubel et al. [2014]. In Section 5.1, we extend our results to unbounded distributions, including the unbounded Pareto distributions studied by Wang and Zender [2002], Federico and Rahman [2003], and Holmberg [2009]; our sufficient condition remains satisfied for unbounded Pareto distributions.

[38] For any $\varphi(p)$, $v(\varphi(p)) - p \searrow 0$ as $n \nearrow \infty$.

ing—irrespective of the marginal value function v.[39] This follows since the numerator on the right is negative (the marginal values are decreasing and thus $v_q < 0$) while the denominator is positive (since v is decreasing and the support of the weighting distribution is above Q for $Q < \overline{Q}$). In the sequel we illustrate our other results with additional examples in which a pure-strategy equilibrium exists.[40]

While our sufficient condition shows that the equilibrium exists in many cases of interest, there are situations in which the equilibrium does not exist; see the discussion in our introduction.[41]

Our next step is to establish that the equilibrium is unique.

Theorem 2. [Uniqueness] *The Bayesian-Nash equilibrium is unique.*

Two important comments on this and subsequent results are in order:

- We state this and all our subsequent results presuming that the bidder's marginal utility v and the supply distribution F are such that the equilibrium exists.
- In this result and all subsequent results we restrict attention to bids at relevant quantities: that is, quantities that an agent has positive probability of winning. Bids for the quantities that the bidder never wins have to be weakly decreasing and sufficiently competitive, but they are not determined uniquely.[42]

The existence and uniqueness results lead us to our main insight, the bid representation theorem:

[39]The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg [2009].

[40]The first of these examples features a normal distribution, the second one features strictly concave marginal values, and the last one features reserve prices. In general, our existence condition is closed with respect to several changes of the environment: adding a bidder preserves existence, making the marginal values less concave (or more convex) preserves existence, and imposing a reserve price preserves existence.

[41]The construction of a tighter existence condition is complicated by the possibility of monotonicity-constrained deviations from the symmetric solution to the market clearing equation provided in Theorem 3. A global best response might exist which is the aggregation of nonoptimal local behavior.

[42]The reason a bidder's bids on never-won quantities need to be sufficiently competitive is to ensure that other bidders do not want decrease their bids on relevant quantities. In the setting with reserve prices, which we analyze in Section 5, the bids on never-won quantities may not need to be competitive and hence these bids are even less determined, but the equilibrium bids on the relevant quantities, that is, those which are sometimes marginal, remain uniquely determined. Importantly, these bids being insufficiently competitive does not induce alternate equilibria: there are no equilibria in which these bids are lower than required to support the unique equilibrium we find.

Theorem 3. [Bid Representation Theorem] *In the unique equilibrium, bids are given by*

$$b\left(q\right)=\int_{nq}^{\overline{Q}}v\left(\frac{x}{n}\right)dF^{nq,n}\left(x\right). \tag{1}$$

The resulting market price function is given by

$$p\left(Q\right)=\int_{Q}^{\overline{Q}}v\left(\frac{x}{n}\right)dF^{Q,n}\left(x\right). \tag{2}$$

Thus, the equilibrium price p is the appropriately-weighted average of bidders' marginal values v, and the same applies to equilibrium bid functions.

Consider three examples. Substitution into our bid representation shows that when marginal values v are linear and the supply distribution F is generalized Pareto, $F\left(x\right)=1-\left(1-\frac{x}{\overline{Q}}\right)^{\alpha}$ for some $\alpha>0$, the equilibrium bids are linear in quantity. This case of our general setting has been analyzed by Ausubel et al. [2014].[43] Our bid representation remains valid when F puts all its mass on $\overline{Q}$: taking the limit of continuous probability distributions which place increasingly more probability near $\overline{Q}$, the representation implies that equilibrium bids are flat, as they should be. Finally, Figure 1 illustrates the equilibrium bids for ten bidders with linear marginal values who face a distribution of supply that is truncated normal. This and the subsequent figures represent bids, marginal values, and the c.d.f. of supply; it is easy to distinguish between the three curves since bids and the marginal values are decreasing (and bids are below marginal values) while the c.d.f. is increasing.[44]

The above three theorems are proved in the Appendix. The rest of our paper builds upon them to establish qualitative properties of the unique equilibrium, and to provide guidance as to how to design divisible good auctions.

[43] Ausubel et al. [2014] calculated bid functions in terms of the parameters of their model (linear marginal values and Pareto distribution of supply) and do not rely on or recognize the separability property that is crucial to our analysis. While we focus on bounded distributions, Ausubel et al. [2014] look at both bounded and unbounded Pareto distributions, and Wang and Zender [2002] look at unbounded Pareto distributions. Our general approach remains valid for unbounded distributions, including Pareto, except that uniqueness requires a lower bound on admissible bids, e.g. an assumption that bids are nonnegative. We provide more details on this extension of our results in our discussion of reserve prices.

[44] In all figures, we check our equilibrium existence condition and calculate bids numerically using R. In Figure 1 we use a normal distribution with mean 3 and standard deviation 1, truncated to the interval $[0,6]$.

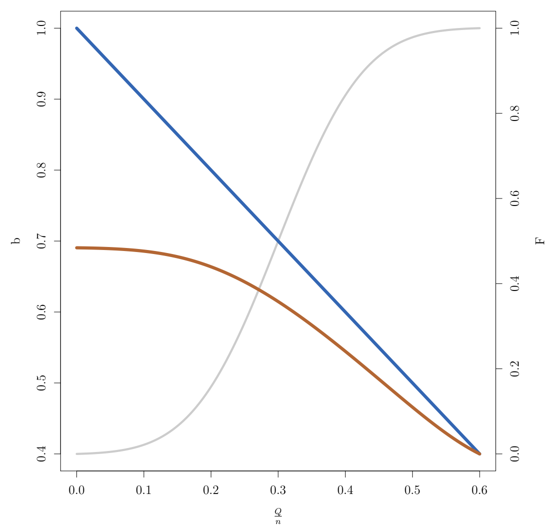

Figure 1: Equilibrium bids when the distribution of supply Q is truncated normal.

4 Properties of Equilibrium and Comparative Statics

Let us start by recognizing some immediate corollaries of the bid representation theorem. While we present these results as corollaries, parts of Corollaries 1 and 3 are among the key lemmas in the proofs of the main results of the previous section. In all such cases we provide direct proofs of the relevant results in the Appendix.

Corollary 1. [Properties of Equilibrium] *The unique equilibrium is symmetric, and its bid functions are strictly decreasing and differentiable in quantities.*

Recall that we impose no assumptions on symmetry of equilibrium bids, their strict monotonicity, nor continuity; we derive these properties.

Since the unique equilibrium is symmetric, there is an easy correspondence between the market price $p(Q)$ given supply Q and the bid functions $b^i = b$:

$$b(q) = p(nq).$$

This relationship is embedded in the bid representation theorem.

4.1 Flat Bids, Low Margins, and Concentrated Distributions

A case of particular interest arises when the distribution of supply is concentrated near some target quantity. We say that a distribution is δ-concentrated near quantity Q^* if

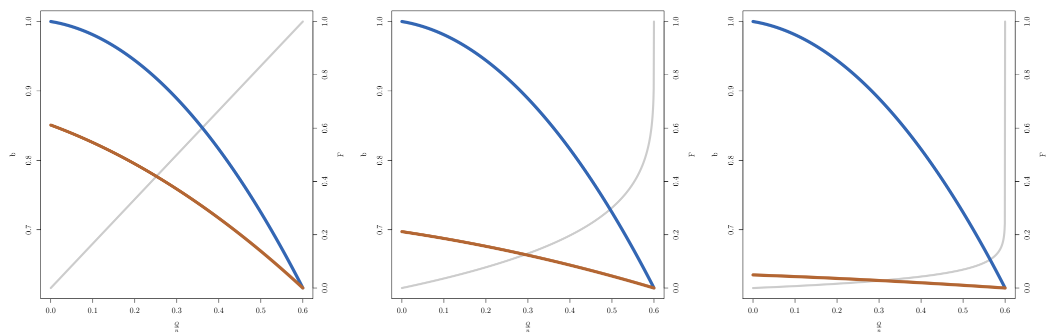

Figure 2: Bids are flatter for more concentrated distributions of supply.

$1-\delta$ of the mass of supply is within δ of quantity Q^*.

Our bid representation theorem implies that the bids on initial quantities are nearly flat for concentrated distributions.

Corollary 2. [Flat Bids] *For any $\varepsilon > 0$ and quantity Q^* there exists $\delta > 0$ such that, if the supply is δ-concentrated near Q^*, then the equilibrium bids for all quantities lower than $\frac{Q^*}{n} - \varepsilon$ are within ε of $v\left(\frac{Q^*}{n}\right)$.*

Figure 2 illustrates the flattening of equilibrium bids; in the three sub-figures ten bidders face supply distributions that are increasingly concentrated around the total supply of 6 (per capita supply of 0.6).

Our representation theorem has also implications for bidders' margins. In the corollary below we refer to the supremum of quantities the bidder wins with positive probability as the highest quantity a bidder can win in equilibrium.

Corollary 3. [Low Margins] *The highest quantity a bidder can win in equilibrium is $\frac{1}{n}\overline{Q}$, and the bid at this quantity equals the marginal value, $b\left(\frac{1}{n}\overline{Q}\right) = v\left(\frac{1}{n}\overline{Q}\right)$. Furthermore, for any $\varepsilon > 0$ and quantity Q^* there exists $\delta > 0$ such that, if supply is δ-concentrated near Q^*, then each bidder's equilibrium margin $v\left(\frac{1}{n}Q^* - \delta\right) - b\left(\frac{1}{n}Q^* - \delta\right)$ on the $\frac{1}{n}Q^* - \delta$ unit is lower than ε.*

Thus, each bidder's margin on the last unit they could win is zero; and, if the supply is concentrated around some quantity Q^*, then the margin on units just below $\frac{1}{n}Q^*$ is close to zero.

4.2 Comparative Statics

Our bid representation theorem allows us to easily deduce how bidding behavior changes when the environment changes. First, as one could expect, an increase in marginal values always benefits the seller: higher values imply higher revenue.

Corollary 4. [Higher Values] *If bidders' marginal values increase, the seller's revenue goes up.*

The bid representation theorem further implies that if there is an affine transformation of bidders' marginal values from v to $\alpha v + \beta$, then the seller's revenue changes from π to $\alpha\pi + \beta$. In particular, all the additional surplus goes to the seller when the value of all bidders is raised by a constant.

Also as one would expect, the bidders' equilibrium margins are lower and the seller's revenue is larger when there are more bidders and the distribution of supply is held constant:

Corollary 5. [More Bidders] *The bidders submit higher bids and the seller's revenue is larger when there are more bidders (both when the supply distribution is held constant, and when the per-capita supply distribution is held constant).*

Indeed, as the number of bidders increases, $1 - F^{Q,n}(x) = \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}}$ decreases, and hence $F^{Q,n}(x)$ increases, thus the mass is shifted towards lower x, where marginal values are higher. At the same time, the marginal value at x either increases in n (if we keep the distribution of supply constant) or stays constant (if we keep the distribution of per-capita supply constant). Both these effects point in the same direction, implying that the bids and expected revenue increase in the number of bidders. This argument also shows that the seller's revenue is increasing if we add bidders while proportionately raising supply, and that a bidder's profits are decreasing in the number of bidders even if we keep per-capita supply constant.

While bidders raise their bids when facing more bidders even if the per-capita distribution stays constant, our bid representation theorem implies that the changes are small.[45] This is illustrated in Figure 3 in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 *million* bidders. To see analytically what happens for large numbers

[45] Notice that if we keep the supply distribution fixed while more and more bidders participate in the auction, then in the large market limit the revenue converges to average supply times the value on the initial unit. See Swinkels [2001].

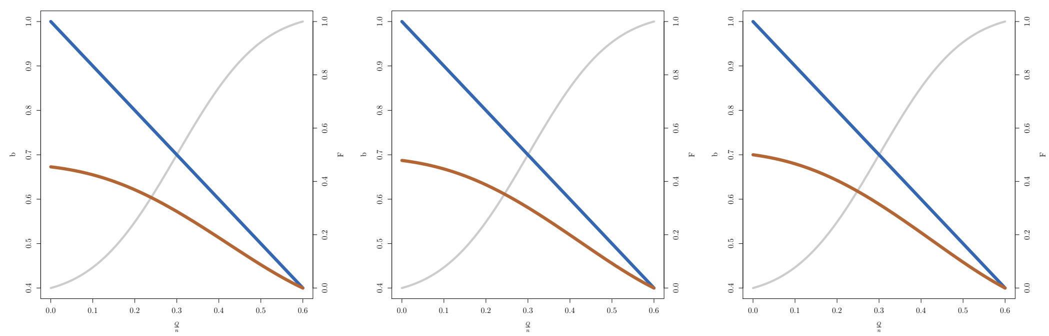

Figure 3: Bids go up when more bidders arrive (and per capita quantity is kept constant) but not by much: 5 bidders on the left, 10 bidders in the middle, and 5 million bidders on the right. Note that all axis scales are identical.

of bidders, let us denote the distribution of per-capita supply by $\overline{\overline{F}}$,

$$\overline{F}\left(x\right)=F\left(nx\right),$$
$$\overline{F}^{q,n}\left(x\right)=F^{nq,n}\left(nx\right).$$

and note that $b\left(q\right)=\frac{1}{n}\int_{q}^{\frac{1}{n}\overline{Q}}v\left(x\right)d\overline{F}^{q,n}\left(x\right)$. As $n\rightarrow\infty$ we get $\overline{F}^{q,n}\left(x\right)\rightarrow\frac{\overline{F}(x)-\overline{F}(q)}{1-\overline{F}(q)}$, and the limit bids take the form

$$b\left(q\right)=\frac{1}{n}\int_{q}^{\frac{1}{n}\overline{Q}}v\left(x\right)d\frac{\overline{F}\left(x\right)-\overline{F}\left(q\right)}{1-\overline{F}\left(q\right)}=\frac{1}{n}\left(\frac{1}{1-\overline{F}\left(q\right)}\right)\int_{q}^{\frac{1}{n}\overline{Q}}v\left(x\right)d\overline{F}\left(x\right).$$

In particular, in large markets, the bid for any unit is given by the average marginal value of higher units, where the average is taken with respect to (scaled) per-capita supply distribution.

5 Designing Pay-as-Bid Auctions

The reserve price and the distribution of supply are two natural elements of pay-as-bid auction that the seller can select.[46] We now leverage our bid representation theorem to analyze these design choices. In the process we relax the assumption that the distribution

[46] In discussing the design problem we will maintain the assumption that the pay-as-bid format is run. The optimal mechanism design for selling divisible goods has been analyzed by Maskin and Riley [1989]; the optimal mechanism is complex and it is not used in practice. In addition to setting supply and reserve prices, the choice in practice is between pay-as-bid and uniform price auctions. We address the latter question in the next section.

of supply is bounded. Since design decisions are taken from the seller's perspective, we reintroduce bidder information into our terminology.

5.1 Reserve Prices

We first consider the case in which $v(q;s)$ does not depend on the bidders' signal s, or equivalently the seller knows the bidders' signal. A key step of the analysis of this special case extends our characterization of the equilibrium of the bidders' game to the pay-as-bid with a reserve price. This extension play the key role in the general case, which consider next.

5.1.1 Complete Information

Our analysis of reserve prices when the seller knows the values is based on the following:

Theorem 4. [Equilibrium with Reserve Prices] *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. In the pay-as-bid auction with reserve price R, the equilibrium is unique and is identical to the unique equilibrium in the pay-as-bid auction with supply distribution $F^R(Q) = F(Q)$ for $Q < \hat{Q}$ and $F^R\left(\hat{Q}\right) = 1$, where $\hat{Q} = nv^{-1}(R)$.*

Notice that distribution F^R has a probability mass at supply $\hat{Q}$, which is the largest supply under this distribution. While we derived our results for atomless distributions, our arguments would not change if we allowed an atom at the highest supply. Thus, all our equilibrium results remain applicable.[47]

The rest of the proof of Theorem 4 is then simple. When the distribution of supply is F^R then the last relevant bid is exactly R by Corollary 3, and hence imposing the reserve price of R does not change bidders' behavior. Furthermore the equilibrium bids against F^R remain equilibrium bids against F with reserve price R, and one direction of the Theorem is proven. Consider now equilibrium bids in an auction with reserve price R.[48] The sum of bidders' demands is then always weakly lower than $nv^{-1}(R)$ and

[47]We provide more details in the Appendix. Importantly, if marginal values v and a distribution of supply F satisfy our sufficient condition for equilibrium existence, then F^R satisfies this condition as well. Indeed, for $Q < \hat{Q}$, folding the tail of the distribution F into an atom in F^R leaves the left-hand side of this condition unchanged while making the right-hand side more negative (since its numerator is negative and the mass shift makes the positive denominator smaller).

[48]Instead of this step of the argument, we could check directly that our uniqueness result, Theorem 4, remains true in the setting with reserve prices.

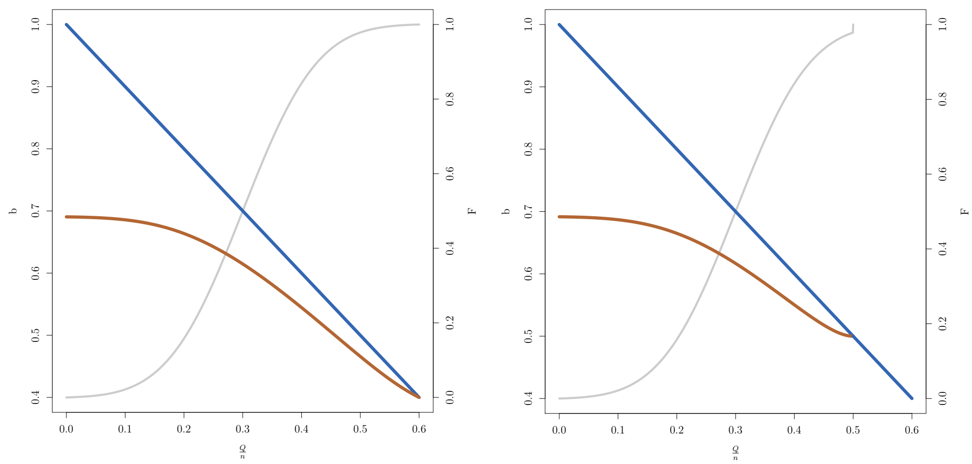

Figure 4: The equilibrium bid function with normal distribution of supply (left), with optimal reserve price (right). The bid for the implicit “maximum quantity” equals the marginal value for this quantity, and the entire bid function shifts up.

hence their bids constitute an equilibrium when the supply is distributed according to F^R. This establishes the other direction of Theorem 4.

An immediate corollary from the equivalence between reserve prices and a particular change in supply distribution is:

Corollary 6. [Reserve Price as Supply Restriction] *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. For every reserve price R there is a reduction of supply that is revenue equivalent to imposing R.*

Without bidder information all reserve prices can be mimicked by supply decisions, but not all supply decisions can be mimicked by the choice of reserve prices. In particular, the revenue with optimal supply is typically higher than the revenue with optimal reserve price. Notice also that, with concentrated distributions, our results imply that attracting an additional bidder is more profitable then setting the reserve price right.

Our analysis of optimum supply in the next subsection further implies that:

Corollary 7. [Optimal Reserve Price] *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. The optimal reserve price R is equal to bidders' marginal value at the optimal deterministic supply: $R \in \max_{R'} R'v^{-1}(R')$.*

When the reserve price R is binding, the equivalence between reserve prices and supply restrictions gives an implicit “maximum supply” of $\overline{Q}^R = nv^{-1}(R)$. At this quantity, parceled over each agent, each agent's bid will equal her marginal value, as at $\overline{Q}$ in the unrestricted case. Since bids fall below values, this bid is weakly above the bid placed at this quantity when there is no reserve price. For quantities below $\overline{Q}^R$ the c.d.f. is unchanged, hence our representation and uniqueness theorems combine to imply that the bids submitted with a reserve price will be higher than without. These effects can be seen in Figure 4.

To conclude this subsection discussion of reserve prices let us notice that we developed the theory of equilibrium bidding assuming that the distribution of supply is bounded. However, in the presence of a reserve price, any unbounded distribution is effectively bounded, hence the boundedness assumption may be relaxed.

5.1.2 Incomplete Information

Now consider the general case in which the seller does not know the bidders' signal s, but has a belief about it, $s \sim \sigma$. The key insights of our analysis do not hinge on any assumptions on the distribution σ nor on $v(q;\cdot)$ as long as the latter is integrable; for exposition's sake it is useful to assume that $v(q;\cdot)$ is well-behaved.

In the general case, both reserve price and quantity restriction plays a role in optimizing pay-as-bid. To see it, consider the case of deterministic supply as our analysis of optimal supply establishes the key insight (Theorem 6) that the optimal supply is deterministic regardless of the reserve price. With reserve price R and deterministic supply Q the revenue is

$$\begin{aligned}\mathbb{E}_s[\pi] = &\Pr\left(v\left(\frac{Q}{n};s\right) \geq R\right)\mathbb{E}\left[v\left(\frac{Q}{n};s\right)|v\left(\frac{Q}{n};s\right) \geq R\right]Q \\ &+ \Pr\left(v\left(\frac{Q}{n};s\right) < R\right)R\mathbb{E}\left[n\varphi(\cdot;s)(R)|v\left(\frac{Q}{n};s\right) < R\right],\end{aligned}$$

where $\varphi(\cdot;s)$ is the inverse of $v(\cdot;s)$. If we assume that $v(\cdot;s)$ is monotonically increasing, given a quantity Q and a reserve price R there is a threshold type $\tau(Q,R)$ such that $v(Q;s) \geq R$ for all $s > \tau(Q,R)$ and $v(Q;s) \leq R$ for all $s < \tau(Q,R)$; equivalently, $\varphi(R;s) \geq Q$ for all $s > \tau(Q,R)$ and $\varphi(R;s) \leq Q$ for all $s < \tau(Q,R)$. Expected revenue can then be expressed as a sum over two integrals,[49]

$$\mathbb{E}_s[\pi] = \int_{-\infty}^{\tau(Q,R)} nR\varphi(R;s)\,d\sigma(s) + \int_{\tau(Q,R)}^{+\infty} Qv\left(\frac{Q}{n};s\right)d\sigma(s).$$

From this expression, the seller's choice of optimal (deterministic) quantity and reserve price can be found by taking first-order conditions; assuming further that $v(q;\cdot)$ is continuous gives[50]

$$\begin{aligned}\frac{\partial \mathbb{E}_s[\pi]}{\partial R} =& \int_{-\infty}^{\tau(Q,R)} n\varphi(R;s) + nR\varphi_R(R;s)\,d\sigma(s)\\ &+ \frac{\partial \tau}{\partial R}(Q,R)\left[nR\varphi(R;\tau(Q,R))\right] - \frac{\partial \tau}{\partial R}(Q,R)\left[Qv\left(\frac{Q}{n};\right)\right]\\ =& n\int_{-\infty}^{\tau(Q,R)} \frac{\partial}{\partial R}\left[R\varphi(R;s)\right]d\sigma(s).\end{aligned}$$

Similar calculations imply $\partial\mathbb{E}_s[\pi]/\partial Q = \int_{\tau(Q,R)}^{+\infty}(\partial[Qv(Q/n;s)]/\partial Q)d\sigma(s)$. That is, the problem of selecting optimal supply and reserve price is identical to the decoupled problems of maximizing revenue on $s \in (-\infty, \tau(Q,R))$ by setting a price, and maximizing revenue on $s \in (\tau(Q,R), +\infty)$ by setting a quantity. Although the bound $\tau(Q,R)$ on the signal intervals is endogenous to the selection of the optimal Q and R, conditional on this bound Q and R are optimal from the standard monopolist's perspective.

Example 1. Take some constants $\rho, \underline{s}, \overline{s} > 0$, such that $\overline{s} > \underline{s} \geq \rho\overline{Q}/n$ and suppose that s is distributed uniformly on $(\underline{s}, \overline{s})$ and $v(q;s) = s - \rho q$ for some constant $\rho > 0$. Thus, $\varphi(R;s) = (s-R)/\rho$. For every relevant deterministic supply Q and reserve price R is then the unique cut-off $\tau = \tau(Q,R) = R + \rho Q/n$ such that

$$R = v\left(\frac{Q}{n};\tau\right) = \tau - \rho\frac{Q}{n}.$$

For all $s < \tau(Q,R)$ the seller sells quantity $\varphi(R;s) = n(s-R)/\rho$ at price R; for all

[49] If $v(q;\cdot)$ were not monotone, the first integral would be taken over the set $\underline{\mathcal{S}} = \{s : v(Q/n;s) < R\}$ and the second integral would be taken over the set $\overline{\mathcal{S}} = \{s : v(Q/n;s) \geq R\}$. Which of these sets contains the "endpoint" signals s such that $v(Q/n;s) = R$ is not relevant.

[50] If $v(q;\cdot)$ were not continuous, the derivatives with respect to the bounds of integration still would cancel: any signal realizations "lost" in the first integral are necessarily "gained" by the second, and vice-versa. Since the definition of τ implies that $nR\varphi(R;\tau(Q,R)) = Qv(Q/n;\tau(Q,R))$ the integrand-mass associated with the shifting boundaries is equal in both integrals, hence the terms cancel regardless of the well-behavedness of $v(q;\cdot)$.

$s > \tau(Q, R)$ the seller sells quantity Q at price $v(Q/n; s) = s - \rho Q/n$. The seller's two-part maximization problem is then[51]

$$\begin{aligned}\max_{Q,R} &\left(\frac{\overline{s} - \tau(Q,R)}{\overline{s} - \underline{s}}\right) \mathbb{E}_s \left[\left(s - \frac{\rho Q}{n}\right) Q | s > \tau(Q,R)\right] \\ &+ \left(\frac{\tau(Q,R) - \underline{s}}{\overline{s} - \underline{s}}\right) \mathbb{E}_s \left[n \left(\frac{s - R}{\rho}\right) R | s < \tau(Q,R)\right].\end{aligned}$$

Given the formula for $s^*(Q, R)$, removing the multiplicative constants from the optimization gives

$$\begin{aligned}\max_{Q,R} \quad &\left\{\left(\overline{s} - \left(R + \frac{\rho Q}{n}\right)\right) \left[\left(\frac{1}{2}\left(\overline{s} + \left(R + \frac{\rho Q}{n}\right)\right) - \frac{\rho Q}{n}\right) Q\right]\right. \\ &\left. + \frac{n}{\rho}\left(\left(R + \frac{\rho Q}{n}\right) - \underline{s}\right) \left[\left(\frac{1}{2}\left(\left(R + \frac{\rho Q}{n}\right) + \underline{s}\right) - R\right) R\right]\right\}.\end{aligned}$$

The first-order conditions with respect to Q and R yield[52]

$$\begin{aligned}\frac{\partial}{\partial Q}\mathbb{E}_s[\pi] :\ 0 &= \left(\left(\overline{s} - \frac{\rho Q}{n}\right)^2 - R^2\right) - \frac{2\rho Q}{n}\left(\left(\overline{s} - \frac{\rho Q}{n}\right) - R\right), \\ \frac{\partial}{\partial R}\mathbb{E}_s[\pi] :\ 0 &= \frac{n}{\rho}\left(\left(\frac{\rho Q}{n}\right)^2 - (R - \underline{s})^2\right) - 2RQ.\end{aligned}$$

Note that these are identically $0 = \int_\tau^{\overline{s}} (\partial[Qv(Q/n; s)]/\partial Q) ds$ and $0 = \int_{\underline{s}}^{\tau} (\partial[R\varphi(R; s)]/\partial R) ds$, as given in the previous argument. The first order conditions can be simplified to a linear system,

$$\begin{aligned}\left(\frac{\rho Q}{n}\right) - (R - \underline{s}) - 2R &= 0, \\ \left(\overline{s} - \frac{\rho Q}{n} + R\right) - \frac{2\rho Q}{n} &= 0.\end{aligned}$$

The solution is

$$R^\star = \frac{\overline{s} + 3\underline{s}}{8}, \quad Q^\star = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n.$$

This solution gives the optimal deterministic supply of $Q^\star$ and the optimal reserve price

[51] Since the uniform distribution is massless, we can ignore the event $s = s^*(Q, R)$. Also, for expositional purposes we constrain attention to cases in which the seller's problem has an interior solution.

[52] See Appendix H for detailed calculations of the first-order conditions and subsequent equations.

of $R^\star$ provided $Q^\star \leq \overline{Q}$.[53] The reserve price is binding because $\overline{s} > \underline{s}$ implies that $R^\star > \frac{7\underline{s}-3\overline{s}}{8} = \underline{s} - \rho\frac{Q^\star}{n} = v\left(\frac{Q^\star}{n}; \underline{s}\right)$.

Consider now two alternate problems, one in which a standard monopolist posts a price, and one in which the monopolist commits to a quantity. In the former, the monopolist solves

$$\max_p n\mathbb{E}_s\left[\frac{1}{\rho}(s-p)\,p\right].$$

Then $p^{\text{MONOP}} = (\overline{s}+\underline{s})/4$. In the latter problem, the monopolist solves

$$\max_q \mathbb{E}_s\left[\left(s-\frac{\rho q}{n}\right)q\right].$$

Then $q^{\text{MONOP}} = n(\overline{s}+\underline{s})/4\rho$.

Comparing the monopolist's problems to the pay-as-bid seller's problem, we can see that $p^{\text{MONOP}} > R^\star$ and $q^{\text{MONOP}} < Q^\star$: that is, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem. This feature arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. When reserve price is the only instrument available, the seller needs to balance the desire to extract surplus from high-value consumers against the desire to not sacrifice too much quantity with a too-high reserve price against low-value consumers; in the pay-as-bid auction the high-value consumers "self-discriminate," since their unique bid function exactly equals their marginal value when the quantity for sale is deterministic. When quantity is the only instrument available the seller is still balancing the same forces, but the presence of a reserve price ensures that he will not sacrifice too much surplus to low-value consumers when he sets the quantity relatively high. When values are sufficiently regular this argument generalizes in a natural way.[54]

Theorem 5. [Comparison of Pay-as-Bid Seller to Monopolist] *Let quantity-monopoly profits π^Q be given by $\pi^Q(Q,s) = Qv(Q/n; s)$, and let $\hat{Q}(s) \in \arg\max_q \pi^Q(q,s)$; let price-monopoly profits π^R be given by $\pi^R(R; s) = nR\varphi(R; s)$, and let $\hat{R}(s) \in \arg\max_p \pi^R(p; s)$. Let Q^M be optimal quantity-monopoly supply and R^M be optimal price-monopoly reserve*

[53] If $Q^\star > \overline{Q}$ then $Q = \overline{Q}$ is the optimal supply and the optimal reserve price $R = \frac{\rho\overline{Q}}{3n} + \frac{s}{3}$ is given by the first order condition.

[54] The literature on market regulation has considered whether price or quantity is a better instrument for achieving desired outcomes; the perspective taken is generally that of the regulator, rather than of a monopolist. **?** obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; **?** find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

against $s \sim \sigma$, and let $Q^\star$ and $R^\star$ be the optimal deterministic supply and reserve price from the pay-as-bid seller's problem. If $v(q;\cdot)$ is monotonically increasing for all q,$\pi^Q(\cdot;s)$ is strictly concave for all s and $\hat{Q}(\cdot)$ is monotonically increasing, then $Q^M \leq Q^\star$; if $v(q;\cdot)$ is monotonically increasing for all q, $\pi^R(\cdot;s)$ is strictly concave for all s and $\hat{R}(\cdot)$ is monotonically increasing, then $R^\star \leq R^M$.

Proof. Consider implementing reserve price R; the condition of quantity optimality at $Q^\star(R)$ is

$$0 = \int_{\tau(Q^\star(R),R)}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma(s).$$

Since $\pi(\cdot;s)$ is strictly concave and $\hat{Q}(\cdot)$ is monotonically increasing, for any Q either $\pi_Q^Q(Q;s) < 0$ for all s, or $\pi_Q^Q(Q;s) > 0$ for all s, or there is some $\bar{s}$ such that $\pi_Q^Q(Q;s') \leq 0$ for all $s' > \bar{s}$ and $\pi_Q^Q(Q;s') \geq 0$ for all $s' > \bar{s}$. Neither of the first two cases support the optimality condition above, hence there is $\bar{s} > \tau(Q^\star(R),R)$ such that $\pi_Q^Q(Q^\star(R);s') \leq 0$ for all $s > s'$ and $\pi_Q^Q(Q^\star(R);s') \geq 0$ for all $s < \bar{s}$. Then we have

$$\int_{\tau(Q^\star(R),R)}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) \geq \int_{-\infty}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right).$$

Since $\pi^Q(\cdot;s)$ is strictly concave for all s, whenever $Q < Q^M$, $\pi_Q^Q(Q;s) > \pi_Q^Q(Q^M;s)$. Then if $Q^\star < Q^M$, we have

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) > \int_{-\infty}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^M;s\right)d\sigma\left(s\right).$$

Putting these inequalities together gives

$$0 = \int_{\tau(Q^\star(R);R)}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) > \int_{-\infty}^{+\infty} \frac{\partial}{\partial Q}\pi^Q\left(Q^M;s\right)d\sigma\left(s\right) = 0.$$

This is a contradiction, hence $Q^\star \geq Q^M$.

A similar argument applies to the case of $R^\star \leq R^M$. □

5.2 Optimal Supply

Suppose first that the seller has some quantity $\overline{Q}$ of the good, and would like to design a supply distribution F that maximizes his revenue.[55] For deterministic quantities without a lower bound on prices the problem is simple: offering quantity $\hat{Q} \leq \overline{Q}$ leads to a unique equilibrium in which all bids are flat.[56] The seller's revenue is thus $\mathbb{E}_s[\hat{Q}v\left(\frac{1}{n}\hat{Q};s\right)]$. Let $Q^\star$ be the deterministic monopoly supply; then $Q^\star$ is the quantity that maximizes this expression.

In the presence of a lower bound on prices, the seller must be mindful that the reserve price might affect the quantity sold. In this case, given a reserve price R the seller's expected revenue under a deterministic quantity schedule is $\mathbb{E}_s[\hat{Q}(R;s)v(\hat{Q}(R;s)/n;s)]$.[57] Letting $Q^\star(R)$ be the deterministic monopoly supply given the reserve price R, $Q^\star(R)$ is the quantity that maximizes this expression.[58]

However, the seller has the option to offer a stochastic distribution over multiple quantities, and it is plausible that such randomization could increase his expected revenue. Offering randomization over quantities larger than the optimal deterministic supply $Q^\star(R)$ may be relatively easily shown to be suboptimal: his profit on the units above $Q^\star(R)$ is lower than his profit on deterministically selling $Q^\star(R)$, and moreover offering quantities above $Q^\star(R)$ suppresses the bids submitted for $Q^\star(R)/n$. On the other hand, offering quantities lower than $Q^\star(R)$ offers the seller a trade-off: he sometimes sells less than $Q^\star(R)$, with a direct and negative revenue impact, but when he sells quantity $Q^\star(R)$ he will receive higher payments due to the pay-as-bid nature of the auction.[59]

We answer this question, and show that selling the deterministic supply $Q^\star$ is in fact revenue-maximizing for sellers across all pure-strategy equilibria; for this reason in the sequel we refer to $Q^\star$ as optimal supply.[60]

[55] Under relatively mild conditions—such as, for example, there being a finite $q > 0$ such that for all s, $v(q;s) = 0$.

[56] The standard Bertrand argument suffices. This point was made by Wang and Zender [2002]. Note also that this claim remains true in the presence of reserve prices.

[57] Generally, for any s either $\hat{Q}(R;s) = Q^\star(R)$ or $v(\hat{Q}(R;s)/n;s) = R$ (or both). That is, except for degenerate cases the seller sells some quantity at the reserve price, or sells the full quantity at the bidders' true marginal valuation.

[58] We could also consider selecting a profit-maximizing reserve price conditional on supply; by standard maximization principles this would not affect any of our arguments. Because we are considering the ability to randomize over supply, it is simpler to fix a reserve price.

[59] A priori such trade-offs can go either way; see the Introduction. The problem is well illustrated in Figure 2, in which concentrating supply lowers the bids.

[60] We restrict attention to pure-strategy equilibria. A reason a seller may want to ensure that pure-strategy equilibrium is being played is that the symmetry of equilibrium strategies we proved implies

Theorem 6. [Optimal Supply] *In pure-strategy equilibria, the seller's revenue under non-deterministic supply is strictly lower than her revenue under optimal deterministic supply. Optimal deterministic supply is given by the solution to the monopolist's problem when facing uncertain demand.*

So far we have assumed that the seller has access to quantity $\overline{Q}$ and is free to design the distribution of supply. Our approach can be generalized: if the distribution of supply is exogenously given by F and is not directly controlled by the seller, the revenue maximizing-supply reduction by the seller reduces supply to $Q^{\star}(R)$ whenever the exogenous supply is higher than $Q^{\star}(R)$, and otherwise leaves the supply unchanged.

6 Divisible-good Revenue Equivalence

In practice, sellers of divisible goods are not restricted to running pay-as-bid auctions: the pay-as-bid auction and the uniform-price auction are the two most-commonly implemented auctions in this context. From a practical perspective, which of these two formats is preferred is a highly important question, and has be extensively studied both in the theoretical and empirical literature on divisible good auctions (see the introduction).

The results of the previous section allow us to easily compare the revenues in the two auctions in the case of optimal supply and reserve price: in the uniform price auction the optimal supply is then also $Q^{\star}$. In contrast to pay-as-bid, several equilibria are possible. Among them, the equilibrium in which all bidders bid flat at $v\left(\frac{1}{n}Q^{\star};s\right)$ is revenue-maximizing; if the seller knows the bidders' values, then she can assure that this is the unique equilibrium of the uniform-price auction by setting the reserve price at $v\left(\frac{1}{n}Q^{\star};s\right)$. The revenue from the fully-optimized uniform price auction is then exactly the same as in the pay-as-bid auction.[61]

Theorem 7. [Revenue Equivalence for Divisible-Good Auctions] *Suppose* $v\left(q;s\right) = v\left(q\right)$ *for every quantity* q *and signal* s*. With optimal supply and reserve price, the rev-*

that every pure-strategy equilibrium in pay-as-bid auctions is efficient, while it is immediate to see that mixed-strategy equilibria are not efficient. Since pay-as-bid is largely employed by central banks and governments, the efficiency of allocations may be an important concern. Note also that when considering stochastic supply, we maintain our global restriction to distributions of supply that have strictly positive density on some interval $[0,\overline{Q}']$.

[61] The equivalence of Theorem 7 remains true if the seller is able to set different reserve prices for different units, as then the seller could fully extract bidders' surplus in both auction formats.

enue in the unique equilibrium of the pay-as-bid auction is exactly equal to the revenue in the unique equilibrium of the uniform-price auction.

While the seller's ability to set a reserve price has no impact on the revenue in pay-as-bid with optimal supply when the seller is perfectly informed, it plays an important equilibrium-selection role in uniform price. The above analysis tells us that when the seller does not know the bidders' values, or does not have the ability to set reserve prices, then the two auction formats are revenue equivalent only with respect to the seller-optimal equilibrium of the uniform-price auction. This conclusion remains true when the seller is not perfectly informed.[62]

Theorem 8. [Pay-As-Bid Revenue Dominance] *With optimal supply and reserve price, the revenue in the unique pure-strategy equilibrium of the pay-as-bid auction equals the revenue in the seller-optimal equilibrium of the uniform-price auction. In particular, the revenue in the pay-as-bid auction is always at least as high as the revenue in the uniform-price auction.*

Theorems 7 and 8 suggest an answer to the why the debate over revenue superiority of the two canonical auction formats, pay-as-bid and uniform-price, remains unsettled. As captured by the extensive literature on pay-as-bid and uniform-price auctions, sellers are willing to expend significant energy determining which mechanism is preferable; it is reasonable to assume they are just as interested in the particulars of the mechanism they select. We have shown that if the parameters defining the mechanisms are optimally determined, then the mechanisms are revenue equivalent, hence relatively optimized auctions should have similar revenues, independent of the mechanism employed.[63] And, indeed, this is what the empirical literature finds; see the discussion in the Introduction.

7 Conclusion

We have proved that Bayesian-Nash equilibrium is unique in pay-as-bid auction with symmetrically-informed bidders, provided a sufficient condition for equilibrium existence,

[62] This conclusion remains true also if the seller can only set supply and has no ability to set the reserve price (with essentially the same proof).

[63] Our result looks at symmetric bidders. For asymmetric bidders Ausubel et al. [2014] show that the revenue comparison can go either way. This is reminiscent of the situation in single-good auctions: with symmetry first-price and second-price auctions are revenue equivalent, but this equivalence breaks in the presence of asymmetries.

and stated a surprisingly tractable bid representation. We hope that the tractability of our representation will stimulate future work on this important auction format.

Building on these results, we have discussed the design of pay-as-bid auctions. We have shown that in the pay-as-bid auction optimal reserve prices can always be replicated by supply restrictions; this is a property that substantially differentiates pay-as-bid from uniform-price. We have also shown that optimal supply is deterministic, even when bidders have information unavailable to the seller.

Comparing pay-as-bid to uniform-price, we have established that the two auction formats are revenue-equivalent when the seller knows bidders' values and sets optimally the supply and reserve prices. When the seller does not know bidders' values, optimally designed pay-as-bid weakly dominates uniform-price.

While we have constrained attention to the case of symmetric bidders, we conjecture that these results should provide natural bounds on the bids supplied by asymmetric bidders: when bidders are asymmetric, their bids lie between the bids generated by the symmetric model at the infimum of their values and the symmetric model at the supremum of their values. This may be shown to be the case in models with linear marginal values and constant slopes; the general problem remains open. Verifying this result would provide a natural jumping-off point for development of a theory of bidding behavior with fully-private information.

Our results support the use of pay-as-bid format, and they may explain why pay-as-bid is indeed the most popular format for selling divisible goods such as treasury securities.[64]

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[64] The use of pay-as-bid is also supported by the results in Woodward [2016]. He identifies the strategic ironing effect in pay-as-bid with incomplete information. The presence of this effect suggests that bidders might bid above equilibrium bids in pay-as-bid auctions. Thus the equilibrium-based counterfactual comparisons of revenue performance of pay-as-bid and uniform-price might be consistently biased against pay-as-bid.

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A Auxiliary Lemmas

In this section we present the key lemmas in our results on existence, uniqueness, and bid representation. and their proofs. Because bidder information is known to all bidders it is not relevant to their optimization problems, hence we will revert to the shorthand $v^i(q) = v^i(q; s)$. Let us fix a pure-strategy candidate equilibrium. Recall that bid functions are weakly decreasing and we may assume that they are right-continuous. Given equilibrium bids the market price (that is, the stop-out price) $p(Q)$ is a function of realized supply Q. Our statements are about relevant quantities, that is for each bidder we ignore quantities larger than the maximum quantity this bidder can obtain in equilibrium; for instance in the following lemma, all bidders could submit flat bids above their values for units they never obtain.

Lemma 1. *Bids are below values: $b^i(q) \leq v^i(q)$ for all relevant quantities, and $b^i(q) < v^i(q)$ for $q < \varphi^i\left(p\left(\overline{Q}\right)\right)$.*

Proof. We must first establish that agent i is never subject to "mutual ties." Suppose that there are $q^i_\ell < q^i_r$ and some agent $j \neq i$ with $q^j_\ell < q^j_r$ such that, for all $q^i \in [q^i_\ell, q^i_r]$ and $q^j \in [q^j_\ell, q^j_r]$, $b^i(q^i) = b^j(q^j)$. Let $\bar{q}^i = \mathbb{E}[q^i|p(Q) = b(q^i_r)]$; without loss, we may assume that agent i is such that $\bar{q}^i < q^i_r$. If $v^i(\bar{q}^i) < b^i(q^i_r)$, the agent has a profitable downward deviation. If $v^i(\bar{q}^i) \geq b^i(q^i_r)$, the agent has a profitable upward deviation: she can increase her bid slightly by $\delta > 0$ on $[q^i_\ell, \bar{q}^i)$ (enforcing monotonicity constraints as necessary to the left of q^i_ℓ), and submits her true value function on $[\bar{q}^i, q^i_r]$ (enforcing monotonicity constraints as necessary to the right of q^i_r).[65]

Now suppose that there exists q with $b^i(q) > v^i(q)$; because b^i is monotonic and v^i is continuous, there must exist a range (q_ℓ, q_r) of relevant quantities such that $b^i(q) > v^i(q)$ for all $q \in (q_\ell, q_r)$. The agent wins quantities from this range with positive probability, and hence the agent could profitably deviate to

$$\hat{b}^i(q) = \min\left\{b^i(q), v^i(q)\right\}.$$

Such a deviation never affects how she might be rationed, by the first part of this proof; hence it is necessarily utility-improving.

Now consider $q < \varphi^i\left(p\left(\overline{Q}\right)\right)$. If $b^i(q) = v^i(q)$ then monotonicity of b^i and Lipschitz-continuity of v^i imply that for small $\varepsilon > 0$ winning units $[q, q+\varepsilon)$ brings per unit profit lower than ε. By lowering the bid for this qualities to $\hat{b}^i(q) = \min\left\{b^i(q+\varepsilon), v^i(q+\varepsilon)\right\}$, the utility loss from losing the relevant quantities is at most of the order $\varepsilon^2\left(G_i\left(q+\varepsilon; b^i\right) - G_i\left(q; b^i\right)\right)$. Notice that the probability difference here goes to zero as ε goes to zero (even if there is a probability mass at q). At the same time the cost savings from paying lower bids at quantities higher than $q+\varepsilon$ is of the order ε^2. Hence, this deviation would be profitable. □

Lemma 2. *For no price level p are there two or more bidders who, in equilibrium, bid p flat on some non-trivial intervals of quantities.*

[65] Because we are conditioning on her expected quantity, we do not need to directly consider whether quantities are relevant.

Proof. The proof resembles similar proofs in other auction contexts; we have already established a smaller version of this result in the proof of Lemma 1. Suppose agent i bids p on (q_ℓ, q_r) and bidder j bids p on (q'_ℓ, q'_r). Since the support of supply is $\left[0, \overline{Q}\right]$, it must be that $G^i(q_r; b^i) > G^i(q_\ell; b^i)$ and similarly for bidder j. Now, Lemma 1 implies that by lowering q_r if needed we can assume that $v^i(q_r) < p$ and $q_r < \varphi^i\left(p\left(\overline{Q}\right)\right)$. But then bidder i would gain by raising her bid on (q_ℓ, q_r) by a small ε (and if needed by raising the bids on lower units as little as necessary for her bid function to be weakly decreasing). Indeed, the cost of this increase would go to zero as $\varepsilon \searrow 0$ while the quantity of the good the bidder would gain and the per-unit utility gain would both be bounded away from zero. □

Lemma 3. *The market price $p(Q)$ is strictly decreasing in supply Q.*

Proof. Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some Q, then a small increase in Q will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than Q, contradicting market clearing.

Lemma 2 is sufficient to imply that the market price must be strictly decreasing: at every price level which at least two bidders pay with positive probability for some quantity, at most one of the submitted bid functions is flat. Furthermore, for no price level p that with positive probability a bidder pays for some quantity, we can have exactly one bidder, i, submitting a flat bid at price p on an interval of relevant quantities.[66] Indeed, in equilibrium bidder i cannot benefit by slightly reducing the bid on this entire interval; thus it must be that there is some other agent j whose bid function is right-continuous at price p. If $p = 0$, all opponents $j \neq i$ have a profitable deviation.[67] If $p > 0$, we appeal to Lemma 1. Given that i submits a flat bid and the bids of bidder j are strictly below her values for some non-trivial subset of quantities at which her bid is near p, bidder j can then profit by slightly raising her bid; this reasoning is similar to that given in the proof of Lemma 2. □

Lemma 4. *Bid functions are strictly decreasing.*

[66]We refer to any price level p that with positive probability a bidder pays for some quantity, as a relevant price level.

[67]Here we work in a model in which marginal utilities on all possible units is strictly positive. We could dispense with the strict positivity assumption by allowing negative bids.

Proof. Suppose bidder i bids flat at b in a non-trivial interval (q_l, q_r) of the relevant range of quantities. Because the range is relevant, the probability of the supply being $\overline{Q}$ is zero, Lemma 3 established that the market price is strictly decreasing in supply, we have $b > p\left(\overline{Q}\right)$.[68] Now, if no other bidder bids with positive probability in some interval $[b-\varepsilon, b]$, then bidder i could profitably lower her bid on quantities (q_l, q_r) (and possibly some other quantities above q_r). If there is another bidder who bids with positive probability in $[b-\varepsilon, b]$ for every $\varepsilon > 0$, then by Lemma 1 this bidder earns strictly positive margin on relevant units and she could profitably raise her bid to just above b on the units she bids in $[b-\varepsilon, b]$ for. $\square$

Let us denote $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$; that is, $G^i\left(q; b^i\right)$ is the probability that agent i receives at most quantity q when submitting bid b^i in the equilibrium considered. The monotonicity of bid functions implies that as long as b^i is an equilibrium bid, and given other equilibrium bids, the probability $G^i\left(q; b^i\right)$ depends on b^i only through the value $b^i\left(q\right)$.

We define the derivative of G^i with respect to b as follows. For any q and b^i, the mapping $\mathbb{R} \ni t \mapsto G^i(q; b^i + t)$ is weakly decreasing in t, and hence differentiable almost everywhere. With some abuse of notation, whenever it exists we denote the derivative of this mapping with respect to t by $G^i_b(q; b^i)$.

Lemma 5. *For each agent i and almost every q we have:*

$$G^i_b(q; b^i) = f(q + \sum_{j \neq i} \varphi^j(b^i(q))) \cdot \sum_{j \neq i} \varphi^j_p(b^i(q)).$$

Proof. By definition, $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$. From market clearing, this is

$$\begin{aligned} G^i\left(q; b^i\right) =& \Pr\left(Q \leq q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right)\right) \\ =& F\left(q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right)\right). \end{aligned}$$

[68] In fact, as long as we ration quantities in a monotonic way, even conditional on the zero-probability event that $Q = \overline{Q}$, Lemma 2 implies that agent i would get no quantity from the flat.

Where the demands φ^j of agents $j \neq i$ are differentiable, we have

$$G_b^i\left(q; b^i\right) = f\left(q + \sum_{j\neq i} \varphi^j\left(b^i\left(q\right)\right)\right) \sum_{j\neq i} \varphi_p^j\left(b^i\left(q\right)\right).$$

Since for all j, the demand function φ^j must be differentiable almost everywhere, the result follows. □

Lemma 6. *At points where $G_b^i(q; b^i)$ is well-defined, the first-order conditions for this model are given by*

$$-\left(v\left(q\right) - b^i\left(q\right)\right) G_b^i\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

Equivalently, the first-order condition can be written as

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\left(\frac{d}{db}Q\left(b^i\left(q\right)\right) - \varphi_p^i\left(b^i\left(q\right)\right)\right) = H\left(Q\left(b^i\left(q\right)\right)\right),$$

where $Q(p)$ is the inverse of $p(Q)$.

Proof. The agent's maximization problem is given by

$$\max_b \int_0^Q \int_0^q v\left(x\right) - b\left(x\right) dx dG^i\left(q; b\right).$$

Integrating by parts, we have

$$\max_b -\left[\left(1 - G^i\left(q; b\right)\right) \int_0^q v\left(x\right) - b\left(x\right) dx\right] |_{q=0}^Q + \int_0^Q \left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q; b\right)\right) dq.$$

In the first square bracket term, both multiplicands are bounded for $q \in [0, Q]$, hence the fact that $1 - G^i(Q; b) = 0$ for all b and $\int_0^0 v(x) - b(x)dx = 0$ for all b allows us to reduce the agent's optimization problem to

$$\max_b \int_0^Q \left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q; b\right)\right) dq.$$

Calculus of variations gives us the necessary condition

$$-\left(1 - G^i\left(q; b^i\right)\right) - \left(v\left(q\right) - b^i\left(q\right)\right) G_b^i\left(q; b^i\right) = 0.$$

This holds at almost all points at which G_b^i is well-defined. Rearrangement yields the first expression for the first-order condition.

To derive the second expression, let us substitute into the above formula for G^i and G_b^i from the preceding lemma. We obtain

$$-\left(v\left(q\right)-b^i\left(q\right)\right)f\left(q+\sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)\left(\sum_{j\neq i}\varphi_p^j\left(b^i\left(q\right)\right)\right)=1-F\left(q+\sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right),$$

Now, $Q\left(p\right)$ is well defined since we shown that p is strictly monotonic. By Lemma 4 the bids are strictly monotonic in quantities and hence $q+\sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)=Q\left(b^i\left(q\right)\right)$, and

$$-\left(v\left(q\right)-b^i\left(q\right)\right)\left(\sum_{j\neq i}\varphi_p^j\left(b^i\left(q\right)\right)\right)=H\left(Q\left(b^i\left(q\right)\right)\right).$$

Since $\sum_{j\neq i}\varphi_p^j\left(b^i\left(q\right)\right)=\frac{d}{db}Q\left(b^i\left(q\right)\right)-\varphi_p^i\left(b^i\left(q\right)\right)$, the second expression for the first order condition obtains. $\square$

Lemma 7. *The market-clearing price for the maximum possible quantity, $p(\overline{Q})$, is uniquely determined. The equilibrium quantities $q^i(\overline{Q})$ are also uniquely determined.*

Proof. We tackle this in two steps: first, if bid functions have finite slope for individual agents' maximum quantities $\overline{q}^i$, then bids meet values at $\overline{q}^i$; second, this must be the case. Although out of proper logical order, the former is simpler to demonstrate than the latter.

First, suppose that in a particular equilibrium each agent's bid function has a finite slope at the maximum-obtainable quantity $\overline{q}^i$: there is $M\in\mathbb{R}_+$ such that $\limsup_{q\nearrow\overline{q}^i}(b^i(q)-b^i(\overline{q}^i)/(\overline{q}^i-q)<M$. This implies that $\liminf_{p\searrow\underline{b}^i}(\varphi^i(\underline{b}^i)-\varphi^i(p))/(p-\underline{b}^i)>1/M$, where $\underline{b}^i=b^i(\overline{q}^i)$. Without loss we can assume that b^i is left-continuous at $\overline{q}^i$. We take the limit of the agent's first-order condition as $q\nearrow\overline{q}^i$, to allow for the fact that only needs to hold almost everywhere. Then we have

$$\lim_{q\nearrow\overline{q}^i}b^i\left(q\right)=\lim_{q\nearrow\overline{q}^i}\left[v^i\left(q\right)+H\left(\sum_j\varphi^j\left(b^i\left(q\right)\right)\right)\left(\sum_{j\neq i}\varphi_p^j\left(b^i\left(q\right)\right)\right)^{-1}\right].$$

As $q\nearrow\overline{q}^i$, $\sum_j\varphi^j(b^i(q))\to\overline{Q}$. Since $f(Q)>0$ everywhere, $H(\overline{Q})=0$; as $\liminf_{p\searrow\underline{b}^i}(\varphi^i(\underline{b}^i)-\varphi^i(b))/(p-\underline{b}^i)$ is bounded away from zero, it follows that $b^i(\overline{q}^i)=v(\overline{q}^i)$.

Note that the above argument is valid as long as at least two agents' bid functions have finite slope at $\overline{q}^i$, as the limit infimum will then be bounded away from zero for all agents. If only one agent's bid function has finite slope at $\overline{q}^i$, then as demonstrated above all other agents $j \neq i$ must have $v(\overline{q}^j) = b^j(\overline{q}^j)$, and by assumption the slope of their bid functions is infinite. But since v is Lipschitz continuous, this implies that $b^j(q) > v(q)$ for q near $\overline{q}^j$, which cannot occur. Thus the only other conditions is that *all* agents' bid functions have infinite slope at $\overline{q}^i$, and again by Lipschitz continuity this requires that $v(\overline{q}^i) > b^i(\overline{q}^i)$ for all i.[69]

It is helpful here to state our proof approach. We first posit a deviation for agent i which "kinks out" her bid function, and renders it flat to the right of some q near $\overline{q}^i$. We then state an incentive compatibility condition which must be satisfied, since in equilibrium this deviation cannot yield a utility improvement. We notice that the ratio of additional bid to profit margin can be made arbitrarily small over the relevant regions of this new flat. Fixing a small deviation, we pick the agent who has the least probability of obtaining quantities affected by the deviation, and then find even smaller deviations for all other agents such that, ceteris paribus, they have an equal probability of their outcome being affected. Since these deviations cannot be profitable in equilibrium, we obtain an inequality which must be satisfied by the sum of all agents' incentives. We then find that, for sufficiently small deviations, this inequality cannot be satisfied, implying that for some agent a small deviation must be profitable. It follows that it cannot be the case that all agents have bids below values.

In this case, for a given agent i and $\delta > 0$, consider a deviation $\hat{b}^i(\cdot;\delta)$ defined by

$$\hat{b}^i(q;\delta) = \begin{cases} b^i(q) & \text{if } q \leq \overline{q}^i - \delta, \\ \min\left\{b^i\left(\overline{q}^i - \delta\right), v(q)\right\} & \text{otherwise.} \end{cases}$$

For small δ, this deviation will give the agent the full marginal market quantity for all realizations $Q \geq \sum_j \varphi^j(b^i(\overline{q}^i - \delta))$. Given this deviation, let $q^\star(q;\delta)$ be the quantity obtained under the deviation when, under the original strategy, the quantity would have

[69] In what follows, we assume that all agents receive positive quantities with positive probability. When this is not the case, Lemma 4 is sufficient to imply that at least two agents receive positive quantities with positive probability, and all subsequent arguments go through when we restrict attention only to such agents.

been $q \geq \overline{q}^i - \delta$. Explicitly,

$$q^\star(q;\delta) = \sum_j \varphi^j\left(b^i(q)\right) - \sum_{j\neq i}\varphi^j\left(\hat{b}^i(q;\delta)\right).$$

We will use this quantity to analyze the additional quantity the deviation yields above baseline $\overline{q}^i - \delta$,

$$\Delta_L^i(q;\delta) \equiv q - \left[\overline{q}^i - \delta\right], \quad \Delta_R^i(q;\delta) \equiv q^\star(q;\delta) - q, \quad \Delta^i(q;\delta) \equiv \Delta_L^i(q;\delta) + \Delta_R^i(q;\delta).$$

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

$$\begin{aligned}
&\int_{\overline{q}^i-\delta}^{\overline{q}^i}\int_{\overline{q}^i-\delta}^{q} \hat{b}^i(x;\delta) - b^i(x)\,dx dG^i\left(q;b^i\right)\\
&\geq \int_{\overline{q}^i-\delta}^{\overline{q}^i}\int_{q}^{q^\star(q;\delta)} v(x) - \hat{b}^i(x;\delta)\,dx dG^i\left(q;b^i\right),\\
\implies &\int_{\overline{q}^i-\delta}^{\overline{q}^i}\int_{\overline{q}^i-\delta}^{q} b^i\left(\overline{q}^i-\delta\right) - b^i\left(\overline{q}^i\right) dx dG^i\left(q;b^i\right)\\
&> \int_{\overline{q}^i-\delta}^{\overline{q}^i}\int_{q}^{q^\star(q;\delta)} v\left(q^\star\left(\overline{q}^i;\delta\right)\right) - b^i\left(\overline{q}^i-\delta\right) dx dG^i(q;\delta).
\end{aligned}$$

Because in the latter expression the inner integrands are constant, we can express this in terms of our $\Delta^i_\cdot$ functions,

$$\begin{aligned}
&\int_{\overline{q}^i-\delta}^{\overline{q}^i} \Delta_L^i(q;\delta)\left(b^i\left(\overline{q}^i-\delta\right) - b^i\left(\overline{q}^i\right)\right) dG^i\left(q;b^i\right)\\
> &\int_{\overline{q}^i-\delta}^{\overline{q}^i} \Delta_R^i(q;\delta)\left(v\left(q^\star\left(\overline{q}^i;\delta\right)\right) - b^i\left(\overline{q}^i-\delta\right)\right) dG^i(q;\delta).
\end{aligned}$$

Since we have assumed that $v^i(\overline{q}^i) > b^i(\overline{q}^i)$ for all i, for all $\kappa > 0$ there is a $\delta^i > 0$ such that for all $\delta' \in (0,\delta^i)$ we have

$$b^i\left(\overline{q}^i-\delta'\right) - b^i\left(\overline{q}^i\right) \leq \kappa\left(v\left(q^\star\left(\overline{q}^i;\delta'\right)\right) - b^i\left(\overline{q}^i-\delta'\right)\right).$$

For any such (κ, δ') we must have

$$\kappa \int_{\overline{q}^i-\delta}^{\overline{q}^i} \Delta_L^i (q;\delta') \, dG^i \left(q; b^i\right) > \int_{\overline{q}^i-\delta}^{\overline{q}^i} \Delta_R^i (q;\delta') \, dG^i \left(q; b^i\right).$$

With a finite number of agents, for any $\kappa > 0$ there is a $\underline{\delta}$ such that for all $\delta' < \underline{\delta}$ the above inequality is satisfied. Picking such a $\underline{\delta}$, let agent i be such that $i \in \arg\max_k G^k(\overline{q}^k - \underline{\delta}; b^k)$, and for each agent j let $\hat{\delta}^j \leq \underline{\delta}$ be defined by

$$G^j \left(\overline{q}^j - \hat{\delta}^j; b^j\right) = G^i \left(\overline{q}^i - \underline{\delta}; b^i\right).$$

Let $Q_{\underline{\delta}} = Q - \sum_j \hat{\delta}^j$, and note that $F(Q_{\underline{\delta}}) = G^j(\overline{q}^j - \hat{\delta}^j)$ for all j. The above incentive compatibility argument must hold for each agent, hence summing over all agents, we have

$$\kappa \sum_j \int_{\overline{q}^j-\hat{\delta}^j}^{\overline{q}^j} \Delta_L^j \left(q; \hat{\delta}^j\right) dG^j \left(q; b^j\right) / F\left(Q_{\underline{\delta}}\right) > \sum_j \int_{\overline{q}^j-\hat{\delta}^j}^{\overline{q}^j} \Delta_R^j \left(q; \hat{\delta}^j\right) dG^j \left(q; b^j\right) / F\left(Q_{\underline{\delta}}\right).$$

Writing this in terms of conditional expectations, this is

$$\begin{aligned} \kappa \sum_j \mathbb{E}\left[\Delta_L^j \left(q; \hat{\delta}^j\right) | Q \geq Q_{\underline{\delta}}\right] &> \sum_j \mathbb{E}\left[\Delta_R^j \left(q; \hat{\delta}^j\right) | Q \geq Q_{\underline{\delta}}\right] \\ &= \sum_j \mathbb{E}\left[\Delta^j \left(q; \hat{\delta}^j\right) | Q \geq Q_{\underline{\delta}}\right] - \mathbb{E}\left[\Delta_L^j \left(q; \hat{\delta}^j\right) | Q \geq Q_{\underline{\delta}}\right]. \end{aligned}$$

By definition, $\Delta_L^j(q; \hat{\delta}^j) = q - [\overline{q}^j - \hat{\delta}^j]$, and it must be that $\Delta^j(q; \hat{\delta}^j) = Q - [\overline{q}^j - \hat{\delta}^j]$. Thus we have

$$(\kappa + 1) \sum_j \mathbb{E}\left[q^j - \left[\overline{q}^j - \hat{\delta}^j\right] | Q \geq Q_{\underline{\delta}}\right] > \sum_j \mathbb{E}\left[Q - \left[\overline{q}^j - \hat{\delta}^j\right] | Q \geq Q_{\underline{\delta}}\right].$$

Since $\sum_j q^j = Q$, this becomes

$$(\kappa + 1) \left(\mathbb{E}\left[Q | Q \geq Q_{\underline{\delta}}\right] - Q_{\underline{\delta}}\right) > n \left(\mathbb{E}\left[Q | Q \geq Q_{\underline{\delta}}\right] - Q_{\underline{\delta}}\right).$$

Dividing through, we have $(\kappa + 1)/n > 1$, and this inequality does not depend on $\underline{\delta}$; since $\kappa > 0$ may be arbitrarily small, this is a contradiction. Thus we cannot have $v(\overline{q}^i) > b^i(\overline{q}^i)$ for all agents i. Having already established that when at least one agent

i has $v(\overline{q}^i) = b^i(\overline{q}^i)$ it must be the case that $v(\overline{q}^j) = b^j(\overline{q}^j)$ for all agents j, we know that all agents submit bid functions which equal their marginal values at their maximum possible quantity.

With $b^i(\overline{q}^i) = v(\overline{q}^i)$ for all i, the remainder of the proof is immediate. By market clearing, we then have $p(\overline{Q}) = v(\overline{q}^i)$ for all i; inverting, we have

$$\overline{Q} = \sum_i v^{-1}\left(p\left(\overline{Q}\right)\right).$$

Since v^i is strictly decreasing in q, there is a unique solution to this equation. From $\overline{q}^i = v^{-1}(p(\overline{Q}))$, it follows that quantities are also unique. $\square$

Lemma 8. *Equilibrium bidding strategies must be symmetric: $b^i = b$ for all i.*

Proof. This proof proceeds by establishing an ordering of asymmetric bid functions. We use this ordering to show that equilibrium is symmetric in the $n = 2$ bidder case. The result from the $n = 2$ bidder case provides tools for the general analysis. Intuitively, these results show that agents do not like receiving zero quantity when it is possible to receive a positive quantity; because this is a necessary feature of asymmetric putative equilibria, these bids are not best responses.

Note that for any agent i, $\sum_{j \neq i} \varphi_p^j(p) = Q_p(p) - \varphi_p^i(p)$. Then we can write the agent's first-order condition as

$$b^i(q) = v(q) + \left(\frac{1 - F(Q(p))}{f(Q(p))}\right)\left(\frac{1}{Q_p(p) - \varphi_p^i(p)}\right).$$

Now suppose that two agents i, j have bid functions which differ on a set of positive measure; without loss, assume that $b^i > b^j$. Then there is a price p such that $\varphi^i(p) > \varphi^j(p)$, and $v(\varphi^i(p)) < v(\varphi^j(p))$. Substituting into the agents' first-order conditions, this gives

$$\left(\frac{1 - F(Q(p))}{f(Q(p))}\right)\left(\frac{1}{Q_p(p) - \varphi_p^i(p)}\right) > \left(\frac{1 - F(Q(p))}{f(Q(p))}\right)\left(\frac{1}{Q_p(p) - \varphi_p^j(p)}\right).$$

Standard rearrangement gives

$$\varphi_p^j(p) < \varphi_p^i(p).$$

Thus whenever $\varphi^i(p) > \varphi^j(p)$, we have $\varphi_p^i(p) > \varphi_p^j(p)$. Recalling from Lemma 7 that bids must equal values at $q = \overline{Q}/n$, this implies that if there is any p such that $\varphi^i(p) > \varphi^j(p)$, then $\varphi^i > \varphi^j$.

Now consider the implications for the $n = 2$ bidder case. Assume that there is p with $\varphi^1(p) > \varphi^2(p) > 0$. Then there is some $\check{p}$ such that $\varphi^2(\check{p}) = 0$ and $\varphi^1(\check{p}) > 0$. Basic auction logic dictates that bidder 1 can never outbid the maximum bid of bidder 2—$b^1(0) = b^2(0)$—thus it must be that bidder 1's first-order condition does not apply for initial units, and she is submitting a flat bid. That is, $b^1(q)|_{q \leq \varphi^1(\check{p})} = \check{p}$. Now let $\varepsilon, \delta > 0$, and define a deviation $\hat{b}^2$ for bidder 2,

$$\hat{b}^2(q) = \begin{cases} b^2(0) + \delta & \text{if } q \leq \varepsilon, \\ b^2(q) & \text{otherwise.} \end{cases}$$

Then for all $q \in (0, \varepsilon]$, $\hat{b}^2(q) > b^1(q)$, and when the realized quantity is $Q \in (0, \varepsilon]$ bidder 2 wins the entire supply. To bound the additional utility, we see that for ε small bidder 2 gains at least

$$\int_0^{\varepsilon} \left(v(x) - b^2(x)\right) dx \left(F\left(\varphi^1(\check{p})\right) - F(\varepsilon)\right).$$

There is an extra cost paid as well; to bound this cost we will assume that it is paid with probability 1, and this cost is $(b^2(0) + \delta)\varepsilon - \int_0^{\varepsilon} b^2(x)dx$. The deviation $\hat{b}^2$ is profitable if the ratio of benefits to costs is greater than 1, hence we look at

$$\lim_{\delta \searrow 0, \varepsilon \searrow 0} \frac{\int_0^{\varepsilon}(v(x) - b^2(x))\,dx\,(F(\varphi^1(\check{p})) - F(\varepsilon))}{(b^2(0) + \delta)\varepsilon - \int_0^{\varepsilon} b^2(x)\,dx} = \lim_{\varepsilon \searrow 0} \frac{\int_0^{\varepsilon}(v(x) - b^2(x))\,dx\,(F(\varphi^1(\check{p})) - F(\varepsilon))}{b^2(0)\varepsilon - \int_0^{\varepsilon} b^2(x)\,dx}.$$

The numerator and denominator both go to zero as $\varepsilon \searrow 0$; application of l'Hopital's rule gives

$$= \lim_{\varepsilon \searrow 0} \frac{v(0) - b^2(0)}{0} = +\infty.$$

Thus the deviation $\hat{b}^2$ is profitable for some $\varepsilon, \delta > 0$.[70]

Now consider the case of $n \geq 3$ agents. By the previous arguments we know that submitted bid functions can be ranked (as can their inverses), and that at least two agents submit the highest possible bid function. Thus we focus attention on two selected

[70]Implicit here is that $v(0) > b^2(0) = b^1(0)$, but this result is trivial: since bidder 1 is bidding flat to $\varphi^1(\check{p})$, if $v(0) = b^1(0)$ she is obtaining zero surplus on a positive measure of initial units. She would rather cut her bid and lose all of these units with some probability, saving payment for higher units and *gaining* probable gross utility.

bid functions,

$$\begin{aligned}
\varphi^H(p) &\equiv \max\left\{\varphi^i(p)\right\}, \\
\varphi^L(p) &\equiv \max\left\{\varphi^i(p) : \ \varphi^i(p) < \varphi^H(p)\right\}.
\end{aligned}$$

Of course, where submitted bid functions are symmetric φ^L will not be well-defined, but we need only pay attention to the asymmetric case. Lastly, let $m_H \equiv \#\{i : \ \varphi^i = \varphi^H\}$ and $m_L = \#\{i : \ \varphi^i = \varphi^L\}$ be the numbers of agents submitting each bid. As mentioned $m_H \geq 2$, and trivially $m_L \geq 1$; additionally, $m_H + m_L \leq n$. As before, there is $\check{p}$ such that $\varphi^L(\check{p}) = 0$, $\varphi^H(\check{p}) > 0$, and $\varphi^L(p) > 0$ for all $p < \check{p}$. Lemma 4 shows that φ^H must be continuous, hence

$$\lim_{p \searrow \check{p}} (m_H - 1)\,\varphi^H_p(p) = \lim_{p \nearrow \check{p}} (m_H - 1)\,\varphi^H_p(p) + m_L \varphi^L_p(p).$$

One obvious solution is $\lim_{p \nearrow \check{p}} \varphi^L_p(p) = 0$; but since $\varphi^L_p \leq \varphi^H_p \leq 0$ this would imply that bids are unboundedly negative, violating monotonicity constraints. Then we have

$$\lim_{p \searrow \check{p}} \varphi^H_p(p) = \lim_{p \nearrow \check{p}} \varphi^H_p(p) + \frac{m_L}{m_H - 1}\varphi^L_p(p) < 0.$$

Intuitively speaking, the bid function b^H is steeper below $\varphi^H(\check{p})$ than above, and there is a kink at this point. This implies a discontinuity in a bidder L's first-order condition near $q = 0$. For p close to but less than $\check{p}$, the first-order condition is

$$\begin{aligned}
& - \left(v\left(\varphi^L(p)\right) - p\right) f\left(Q(p)\right)\left(m_H \varphi^H_p(p) + (m_L - 1)\,\varphi^L_p(p)\right) - \left(1 - F\left(Q(p)\right)\right) = 0, \\
\Longrightarrow \quad & - \left(v\left(\varphi^L(p)\right) - p\right) f\left(Q(p)\right)\left((m_H - 1)\,\varphi^H_p(p) + m_L \varphi^L_p(p)\right) - \left(1 - F\left(Q(p)\right)\right) > 0.
\end{aligned}$$

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Letting $p \nearrow \check{p}$, we know that the term $[(m_H - 1)\varphi^H_p(p) + m_L \varphi^L_p(p)]$ smoothly[71] approaches $\lim_{p \searrow \check{p}} (m_H - 1)\varphi^H_p(p)$, proportional to the marginal probability gained by a slight increase in bid from b^L near $\check{p}$ to $b^L > \check{p}$. Thus, essentially, the L bidder's second-order conditions are not satisfied near $q = 0$, and this is not an equilibrium. $\square$

[71] Both φ^H and φ^L are continuous, hence $[(m_H - 1)\varphi^H_p + m_L \varphi^L_p]$ and $[m_H \varphi^H_p + (m_L - 1)\varphi^L_p]$ are continuous. This additionally implies that φ^L_p and φ^H_p are continuous.

B Proof of Theorem 2 (Uniqueness)

Proof. From Lemma 6 and market clearing, we know that for all bidders

$$(p(Q) - v(q))\, G_b^i\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

Since Lemma 8 tells us that agents' strategies are symmetric, Lemma 5 allows us to write this as

$$\left(p(Q) - v\left(\frac{Q}{n}\right)\right)(n-1)\,\varphi_p(p(Q)) = H(Q).$$

From market clearing, we know that $p(Q) = b(Q/n)$; hence $p_Q(Q) = b_q(Q/n)/n$. Additionally, standard rules of inverse functions give $\varphi_p(p(Q)) = 1/b_q(Q/n)$ almost everywhere. Thus we have

$$\left(p(Q) - v\left(\frac{Q}{n}\right)\right)\frac{n-1}{n} = H(Q)\, p_Q(Q).$$

Now suppose that there are two solutions, p and $\hat{p}$. From Lemma 7 we know that $p(\overline{Q}) = \hat{p}(\overline{Q})$. Suppose that there is a Q such that $\hat{p}(Q) > p(Q)$; taking Q near the supremum of Q for which this strict inequality obtains we conclude that $\hat{p}_Q(Q) < p_Q(Q)$.[72] But then we have

$$\hat{p}(Q) > p(Q) = v\left(\frac{Q}{n}\right) + \left(\frac{n}{n-1}\right) H(Q)\, p_Q(Q) > v\left(\frac{Q}{n}\right) + \left(\frac{n}{n-1}\right) H(Q)\, \hat{p}_Q(Q).$$

The right-continuity of bids, and hence of p, allows us to conclude that if p solves the first-order conditions, $\hat{p}$ cannot. $\square$

C Proof of Theorem 3 (Bid Representation)

From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

$$p_Q = p\tilde{H} - \hat{v}\tilde{H},$$

[72]The inequality inversion here from usual derivative-based approaches reflects the fact that we are "working backward" from $\overline{Q}$, while any solution must be weakly decreasing: thus a small *reduction* in Q should yield $\hat{p}(\overline{Q}) = p(\overline{Q}) \leq p < \hat{p}$.

where $\hat{v}(x) = v(x/n)$, and $\tilde{H}(x) = [1/H(x)][(n-1)/n]$. The solution to this equation has general form

$$p(Q) = Ce^{\int_0^Q \tilde{H}(x)dx} - e^{\int_0^Q \tilde{H}(x)dx} \int_0^Q e^{-\int_0^x \tilde{H}(y)dy} \tilde{H}(x)\,\hat{v}(x)\,dx$$

parametrized by $C \in \mathbb{R}$. Define $\rho = \frac{n-1}{n} \in [\frac{1}{2}, 1)$. We can see that $\tilde{H} = -\rho \frac{d}{dQ} \ln(1-F)$. Thus we have

$$e^{\int_0^t \tilde{H}(x)dx} = e^{-\rho \int_0^t \partial \ln(1-F(x))dx} = e^{-\rho(\ln(1-F(t)) - \ln 1)} = (1 - F(t))^{-\rho}.$$

Substituting and canceling, we have for $Q < \overline{Q}$:

$$p(Q) = \left(C - \rho \int_0^Q f(x)\,(1-F(x))^{\rho-1}\,\hat{v}(x)\,dx\right)(1-F(Q))^{-\rho}. \tag{3}$$

Since $1 - F(\overline{Q}) = 0$, this implies that $C = \rho \int_0^{\overline{Q}} f(x)\,(1-F(x))^{\rho-1}\,\hat{v}(x)\,dx$. The market price is then given by

$$p(Q) = \rho \int_Q^{\overline{Q}} f(x)\,(1-F(x))^{\rho-1}\,\hat{v}(x)\,dx\ \ (1-F(Q))^{-\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$, our formula for market price obtains, and since we have proven earlier that the equilibrium bids are symmetric, the formula for bids obtains as well. This concludes the proof.

D Proof of Theorem 1 (Existence)

While in the main text we present Theorem 1 (existence) as the first result, its proof builds on our Theorems 2 and 3. Of course, the proofs of the latter two theorems do not depend on Theorem 1.

Proof. We want to prove that the candidate equilibrium constructed in Theorem 3 is in fact an equilibrium. Let us this fix a bidder i whose incentives we will analyze, and assume that other bidders follow the strategies of Theorem 3 when bidding on quantities $q \le \overline{Q}/n$ and that they bid $v(\overline{Q}/n)$ for quantities they never win.[73] Since bids and values

[73]When proving the analogue of Theorem 1 in the context of reserve prices, we need to adjust this

are weakly decreasing, in equilibrium there is no incentive for bidder i to obtain any quantity $q > \overline{Q}/n$ and we only need to check that bidder i finds it optimal to submit bids prescribed by Theorem 3 for quantities $q < \overline{Q}/n$. Thus, agent i maximizes

$$\int_0^{\frac{\overline{Q}}{n}} (v(q) - b(q))(1 - G(q; b(q)))\, dq$$

over weakly decreasing functions $b(\cdot)$.

We need to show that the maximizing function $b(\cdot)$ is given by Theorem 3, and because the bid function in Theorem 3 is strictly monotonic, we can ignore the monotonicity constraint. The problem can then be analyzed by pointwise maximization: for each quantity $q \in [0, \overline{Q}/n]$ the agent finds $b(q)$ that maximizes $(v(q) - b(q))(1 - G(q; b(q)))$. Therefore, we can rely on one-dimensional optimization strategies to assert the sufficiency conditions for a maximum. The agent's first-order condition is

$$-\left(1 - G^i(q; b)\right) - (v(q) - b)\, G_b^i(q; b) = 0.$$

Recall that from any symmetric inverse bid of agent i's opponents, $G_b^i(q; b) = (n-1) f(q + (n-1)\varphi(b))\varphi_p(b)$. Then the first-order condition can be expressed as

$$(n-1)(v(q) - b)\varphi_p(b) + Y(q; b) = 0.$$

Suppose that there is $\hat{b}$ that also solves the first-order conditions for the bid for quantity q,[74]

$$(n-1)\left(v(q) - \hat{b}\right)\varphi_p\left(\hat{b}\right) + Y\left(q; \hat{b}\right) = 0.$$

Then since $b(\cdot)$ is continuous and any profitable deviation is such that $\hat{b} \in [b(\overline{Q}/n), b(0)]$ there is some $\hat{q}$ such that $\hat{b} = b(\hat{q})$. At this point,

$$E\left(\hat{q}; \hat{b}\right) = (n-1)\left(v(\hat{q}) - \hat{b}\right)\varphi_p\left(\hat{b}\right) + Y\left(\hat{q}; \hat{b}\right) = 0.$$

If $\partial E/\partial q > 0$ (recall that E is the negative of the first-order condition) whenever

expression to deal with reserve prices: in the analogue, $\overline{Q}$ becomes $nv^{-1}(R)$. The remainder of the argument does not change.

[74] By the assumption of sufficient demand, bidding $\hat{b} = 0$ is never utility-improving. Further, bidding $\hat{b} > b(0)$ is also not utility-improving, so any solution to the first order conditions can be assumed to be internal.

$E(q;b) = 0$ then $E(\cdot;b)$ has a unique zero (if it has any).Then there is at most one solution to the first-order conditions; since the bid representation formula in Theorem 3 gives a closed-form solution for bids and the first-order conditions have a unique solution, the bids given in the representation theorem are an equilibrium. Calculation gives

$$\frac{\partial E}{\partial q} = (n-1)\, v_q\,(q)\,\varphi_p\,(b) + Y_q\,(q;b) > 0.$$

In the symmetric solution to the market clearing equation we have already seen that $(n-1)\varphi_p(b) = Y(\varphi(b))/(b - v(\varphi(b)))$. Substituting this in gives the desired result. □

E Modifying the Proofs to Allow for Reserve Prices

In Section 5 we study reserve prices, and we show that imposing a binding reserve price is equivalent to creating an atom at the quantity at which marginal value equals to the reserve price. In order to extend our results to the setting with reserve prices, we thus need to extend them to distributions in which there might be an atom at the upper bound of support $\overline{Q}$. All our results remain true, and the proofs go through without much change except for the end of the proof of Theorem 3, where more care is needed.

The proof of Theorem 3 goes through until the claim that $1 - F(\overline{Q}) = 0$; in the presence of an atom at $\overline{Q}$ this claim is no longer valid. We thus proceed as follows. We multiply both sides of equation (3) by $(1 - F\,(Q))^{\rho}$ and conclude that

$$p\,(Q)\,(1 - F\,(Q))^{\rho} = C - \rho \int_0^Q f\,(x)\,(1 - F\,(x))^{\rho - 1}\,\hat{v}\,(x)\,dx.$$

Now, let $\overrightarrow{F}(\overline{Q}) \equiv \lim_{Q' \nearrow \overline{Q}} F(Q')$. Because the market price and the right-hand integral are continuous as $Q \nearrow \overline{Q}$, we have

$$p\left(\overline{Q}\right)\left(1 - \overrightarrow{F}\left(\overline{Q}\right)\right) = C - \rho \int_0^{\overline{Q}} f\,(x)\,(1 - F\,(x))^{\rho - 1}\,\hat{v}\,(x)\,dx.$$

The parameter C is determined by this equation. The market price function is then

$$p\,(Q) = \left(\frac{1 - \overrightarrow{F}\left(\overline{Q}\right)}{1 - F\,(Q)}\right)^{\rho} p\left(\overline{Q}\right) + \rho \int_Q^{\overline{Q}} f\,(x)\,(1 - F\,(x))^{\rho - 1}\,\hat{v}\,(x)\,dx\ \ (1 - F\,(Q))^{-\rho}\,.$$

Recall from Lemma 7 that $p(\overline{Q}) = v(\overline{Q}/n)$. Extending our notation to the auxiliary distribution $F^{Q,n}$, we also have

$$F^{Q,n}(\overline{Q}) - \overset{\rightharpoonup}{F}^{Q,n}(\overline{Q}) = 1 - \overset{\rightharpoonup}{F}^{Q,n}(\overline{Q}) = \left(\frac{1-\overset{\rightharpoonup}{F}\left(\overline{Q}\right)}{1-F\left(Q\right)}\right)^{\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$ for all $Q, y < \overline{Q}$, we have

$$\begin{aligned}
p\left(Q\right) &= \left(F^{Q,n}(\overline{Q}) - \overset{\rightharpoonup}{F}^{Q,n}\left(\overline{Q}\right)\right) v\left(\frac{\overline{Q}}{n}\right) + \int_{Q}^{\overline{Q}} v\left(\frac{x}{n}\right) \frac{d}{dy}\left[F^{Q,n}\left(y\right)\right]_{y=x} dx \\
&= \int_{Q}^{\overline{Q}} v\left(\frac{x}{n}\right) dF^{Q,n}\left(x\right),
\end{aligned}$$

proving our formula for equilibrium stop-out price. □

F Proof of Theorem 7 (Optimal Supply)

Suppose R is the reserve price. We allow $R = -\infty$ thus allowing for the case without reserve prices. The market-clearing price is a function of the bidders' information and the reserve price, hence we write $p(Q; R, s)$ instead of $p(Q)$.

Given a reserve price, the revenue is

$$\mathbb{E}\left[\pi\right] = \mathbb{E}_s \int_0^{\overline{Q}} \pi\left(Q; R, s\right) dF\left(Q\right).$$

When bidders' values are relatively low relative to the reserve price, and the realized quantity is high, the bidders only receive a partial allocation. Being mindful of this, expected revenue becomes

$$\mathbb{E}\left[\pi\right] = \mathbb{E}_s \int_0^{\overline{Q}} \int_0^{Q(y;R,s)} p\left(x; R, s\right) dx dF\left(y\right),$$

where $Q(y; R, s)$ is given by

$$Q\left(y; R, s\right) = \begin{cases} y & \text{if } v\left(\frac{y}{n}; s\right) \geq R, \\ nv^{-1}\left(R; s\right) & \text{otherwise.} \end{cases}$$

Integrating by parts gives

$$\mathbb{E}_s\Bigg\{\left[-\left(1-F\left(y\right)\right)\int_0^{Q(y;R,s)} p\left(x;R,s\right)dx\right]\Big|_{y=0}^{\overline{Q}}$$
$$+\int_0^{\overline{Q}}\left(1-F\left(y\right)\right)p\left(Q\left(y;R,s\right);R,s\right)Q_y\left(y;R,s\right)dy\Bigg\}.$$

The first term is zero. By definition, $Q_y(y;R,s)=1$ when $v(y/n;s)\geq R$ and it equals zero otherwise. Then expected revenue can be written as

$$\mathbb{E}_s\int_0^{Q\left(\overline{Q};R,s\right)}\left(1-F\left(y\right)\right)p\left(Q\left(y;R,s\right);R,s\right)dy.$$

For simplicity, we will assume that the distribution of supply is atomless; we are in the process of proving otherwise, but by continuity arguments similar to those in the reserve price calculations everything works in a limiting sense. When this is the case, and if we let $Q^*(s)\equiv Q(\overline{Q};R,s)$,[75] expected revenue can be written as

$$\mathbb{E}_s\int_0^{Q^*(s)}\left(1-F\left(y\right)\right)\left[\left(1-F^{y,n}\left(Q^*\left(s\right)\right)\right)v\left(\frac{Q^*\left(s\right)}{n};s\right)+\int_y^{Q^*(s)}v\left(\frac{x}{n};s\right)dF^{y,n}\left(x\right)\right]dy.$$

Letting $J(Q)=(1-F(Q))^{(n-1)/n}$, this becomes

$$\mathbb{E}\left[\pi\right]=\mathbb{E}_s\int_0^{Q^*(s)}J\left(y\right)^{1/(n-1)}\left[J\left(Q^*\left(s\right)\right)v\left(\frac{Q^*\left(s\right)}{n};s\right)-\int_y^{Q^*(s)}v\left(\frac{x}{n};s\right)J_Q\left(x\right)dx\right]dy.$$

Denote the "pieces" of the integral by T_1 and T_2, so that $\mathbb{E}[\pi]=T_1+T_2$. First,

$$\begin{aligned}T_1&=\mathbb{E}_s\int_0^{Q^*(s)}J\left(y\right)^{1/(n-1)}J\left(Q^*\left(s\right)\right)\hat{v}\left(Q^*\left(s\right);s\right)dy\\&=\mathbb{E}_sJ\left(Q^*\left(s\right)\right)\hat{v}\left(Q^*\left(s\right);s\right)\int_0^{Q^*(s)}J\left(y\right)^{1/(n-1)}dy\\&\leq\mathbb{E}_sJ\left(Q^*\left(s\right)\right)\hat{v}\left(Q^*\left(s\right);s\right)Q^*\left(s\right).\end{aligned}$$

[75]We drop R for brevity since it is fixed and we are looking to show that the optimal distribution of supply is deterministic in the presence of an arbitrary reserve price.

Second,

$$
\begin{aligned}
T_2 &= -\mathbb{E}_s \int_0^{Q^*(s)} J(y)^{1/(n-1)} \int_y^{Q^*(s)} \hat{v}(x;s) J_Q(x)\, dx dy \\
&= -\mathbb{E}_s \int_0^{Q^*(s)} \int_0^{x} J(y)^{1/(n-1)} dy \hat{v}(x;s) J_Q(x)\, dx \\
&\leq -\mathbb{E}_s \int_0^{Q^*(s)} x\hat{v}(x;s) J_Q(x)\, dx \\
&= \mathbb{E}_s \int_0^{Q^*(s)} [x\hat{v}_q(x;s) + \hat{v}(x;s)] J(x)\, dx - Q^*(s)\hat{v}(Q^*(s);s) J(Q^*(s)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbb{E}[\pi] =& T_1 + T_2 \\
\leq& \mathbb{E}_s Q^*(s)\hat{v}(Q^*(s)) J(Q^*(s)) \\
&+ \mathbb{E}_s \int_0^{Q^*(s)} [x\hat{v}_q(x;s) + \hat{v}(x;s)] J(x)\, dx - Q^*(s)\hat{v}(Q^*(s);s) J(Q^*(s)) \\
=& \mathbb{E}_s \int_0^{Q^*(s)} [x\hat{v}_q(x;s) + \hat{v}(x;s)] J(x)\, dx.
\end{aligned}
$$

Notice that $x\hat{v}_q(x;s) + \hat{v}(x;s) = \pi_q^m(x;s)$, where $\pi^m(x;s) = x\hat{v}(x;s)$ is the revenue from selling quantity x at price $\hat{v}(x;s)$. Integrating by parts and denoting

$$Q(x;s) = \min\{x, Q^*(s)\} = \min\{x, Q(\overline{Q}; R, s)\}$$

gives[76]

$$
\begin{aligned}
\mathbb{E}[\pi] &\leq \mathbb{E}_s \int_0^{Q^*(s)} \pi_q^m(x;s) J(x)\, dx \\
&= \mathbb{E}_s \pi^m(Q^*(s);s) J(Q^*(s)) + \int_0^{Q^*(s)} \pi^m(x;s) |J_Q(x)|\, dx \\
&= \mathbb{E}_s \int_0^{\overline{Q}} \pi^m(Q(x;s);s) |J_Q(x)|\, dx.
\end{aligned}
$$

Thus,

$$\mathbb{E}[\pi] \;\; = \;\; \int_0^{\overline{Q}} |J_Q(x)| \mathbb{E}_s [\pi^m(Q(x;s);s)]\, dx.$$

[76]Integration by parts yields $+|J_Q|$ since $J_Q \leq 0$.

Since there are no cross-terms in this integral, the right-hand side is maximized at a degenerate distribution which maximizes $\mathbb{E}_s[\pi^m(Q(x;s);s)]$. But this is exactly the problem of choosing optimal deterministic supply given the reserve price R. It follows that expected revenue is weakly dominated by expected revenue with optimal deterministic supply, hence optimal supply is deterministic. $\square$

G Proof of Theorems 7 and 8 (Revenue Equivalence)

To compare outcomes in the pay-as-bid and uniform-price auctions with optimally-determined supply distributions (following Theorem 6, this supply is deterministic), in the following proof we decorate market outcome functions with superscripts denoting the relevant mechanism. For example, p^{UPA} is the market-clearing price in the uniform-price auction and p^{PABA} is the market-clearing price in the pay-as-bid auction.

Proof. As discussed in Theorem 7, the optimal supply distribution is deterministic in both the pay-as-bid and uniform-price auctions. Revenue maximization may then be expressed as a per-agent quantity $q^\star$ and market price $p^\star$; for signals s such that $v(q^\star;s) \geq p^\star$ it is without loss to assume that the total allocation is $nq^\star$—there is sufficient demand for the total quantity at the reserve price—while for signals s such that $v(q^\star;s) < p^\star$ it is clear that some total quantity $nq' < nq^\star$ will be allocated. The seller's expected revenue is then an expectation over bidder signals,

$$\mathbb{E}_s\left[\pi\right] = \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) \cdot p\left(q^\star, p^\star; s\right)\right].$$

$q^{\text{UPA}}(q^\star, p^\star; s) = q^{\text{PABA}}(q^\star, p^\star; s)$—the quantity allocated under the uniform-price auction equals the quantity allocated under the pay-as-bid auction—-whenever $v(\cdot;s)$ is strictly decreasing at this quantity, or when $v(\cdot;s) > p^\star$ at this quantity.[77] Since we have assumed that $v(\cdot;s)$ is strictly decreasing, the quantity allocation depends only on $q^\star$ and $p^\star$ and not on the mechanism employed. Additionally, it is the case that $p^{\text{UPA}}(q^\star, p^\star; s) =$

[77] In the latter case there is excess demand, so all units will be allocated. In the former case all units are allocated at the reserve price; there is a possible difference in allocation when bidders' marginal values are flat over an interval of quantities at the reserve price, since bidders are indifferent between receiving and not receiving these quantities.

$p^{\text{PABA}}(q^\star, p^\star; s)$ whenever $v(q^\star; s) < p^\star$. Let $\underline{\mathcal{S}}$ be the set of such s,[78]

$$\underline{\mathcal{S}} = \left\{s' : \ v\left(q^\star; s\right) < p^\star\right\}.$$

Then we have

$$\mathbb{E}_s\left[\pi\right] = p^\star \Pr\left(s \in \underline{\mathcal{S}}\right) \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) | s \in \underline{\mathcal{S}}\right] + nq^\star \Pr\left(s \notin \underline{\mathcal{S}}\right) \mathbb{E}_s\left[p\left(q^\star, p^\star; s\right) | s \notin \underline{\mathcal{S}}\right].$$

The left-hand term is independent of the mechanism employed, while the right-hand term depends on the mechanism only via the expected market-clearing price. In the pay-as-bid auction, we have seen that $p(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$, while in the uniform-price auction any price $p \in [p^\star, v(q^\star; s)]$ is supportable in equilibrium. It follows that the pay-as-bid auction weakly revenue dominates the uniform-price auction, and generally will strictly do so. That the seller-optimal equilibrium of the uniform-price auction is revenue-equivalent to the unique equilibrium of the pay-as-bid auction arises from the selection of $p^{\text{UPA}}(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$. □

H Proofs for examples (For Online Publication)

Linear marginal values with generalized Pareto distribution of supply. For generalized Pareto distributions with parameter $\alpha > 0$,

$$\begin{aligned} 1 - F\left(x\right) &= \left(1 - \frac{x}{\overline{\overline{Q}}}\right)^{\alpha}, & f\left(x\right) &= \frac{\alpha}{\overline{\overline{Q}}}\left(1 - \frac{x}{\overline{\overline{Q}}}\right)^{\alpha - 1}; \\ H\left(x\right) &= \frac{1}{\alpha}\left(\overline{Q} - x\right), & H_q\left(x\right) &= -\frac{1}{\alpha}. \end{aligned}$$

Then with linear market values $v(q) = \beta_0 - q\beta_q$,

$$-\frac{1}{\alpha}\left(\overline{Q} - n\varphi\left(p\right)\right)\beta_q + \frac{1}{\alpha}\left(\beta_0 - \varphi\left(p\right)\beta_q - p\right) \propto \beta_0 - \left(\overline{Q} - \left(n-1\right)\varphi\left(p\right)\right)\beta_q - p.$$

For all $Q < \overline{Q}$, $p(Q) > p(\overline{Q})$ and $\overline{Q} > n\varphi(p(Q))$; hence for all $Q < \overline{Q}$,

$$\beta_0 - \left(\overline{Q} - \left(n-1\right)\varphi\left(p\right)\right)\beta_q - p < \beta_0 - \frac{1}{n}\overline{Q}\beta_q - p\left(\overline{Q}\right) = 0.$$

[78]If we constrained attention to monotone $v(q; \cdot)$, we would have $\underline{\mathcal{S}} = [0, \tau)$ for some τ.

Then the existence condition is satisfied for all $Q \in [0, \overline{Q})$.

Linear marginal values with generalized Pareto distribution of supply imply linear bids. Recall our bid representation theorem,

$$b(q) = \int_{nq}^{\overline{Q}} v\left(\frac{x}{n}\right) dF^{nq,n}(x).$$

We integrate by parts to find

$$b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{n} \int_{nq}^{\overline{Q}} 1 - F^{nq,n}(x)\, dx.$$

For generalized Pareto distributions with parameter $\alpha > 0$, we have

$$1 - F^{nq,n}(x) = \left(\frac{\overline{Q} - x}{\overline{Q} - nq}\right)^{\alpha\left(\frac{n-1}{n}\right)}.$$

Integrating, the bid function is

$$b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{\alpha(n-1)+n}\left(\overline{Q} - nq\right).$$

Bids are therefore linear in q.

H.1 Optimal supply and reserve with linear demand (Example 1)

The arguments in Section 5.1.2 demonstrate that optimal supply and reserve price can be found by separately restricting attention to intervals on which the reserve price or the supply restriction are relevant. For completeness's sake we will not use the separation in this argument, and will work through from the joint maximization problem; using the separation argument would allow us to skip the first several steps.

Assuming that Q and R are both binding, which we will subsequently verify, the monopolist's problem is[79]

$$\max_{Q,R} \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s - R) R ds + \int_{\tau}^{\overline{s}} Q\left(s - \frac{\rho Q}{n}\right) ds.$$

[79]Because the signal distribution is uniform, we ignore the constant of proportionality $1/(\overline{s} - \underline{s})$.

Here, $\tau = R + \rho Q/n$. The first-order conditions with respect to Q and R are

$$\begin{aligned}\frac{\partial}{\partial Q}:\ 0 &= \left[\frac{n}{\rho}(\tau - R)R\right]\frac{\partial \tau}{\partial Q} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right]\frac{\partial \tau}{\partial Q} + \int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n} ds,\\ \frac{\partial}{\partial R}:\ 0 &= \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s - 2R)\,ds + \left[\frac{n}{\rho}(\tau - R)R\right]\frac{\partial \tau}{\partial R} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right]\frac{\partial \tau}{\partial R}.\end{aligned}$$

Note that $\tau - R = \rho Q/n$ and $\tau - \rho Q/n = R$; then the $\partial\tau/\partial\cdot$ terms additively cancel, leaving

$$\int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n} ds = 0, \quad \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s - 2R)\,ds = 0.$$

Solving the optimality condition associated with $Q^\star$ gives

$$\frac{1}{2}\left(\overline{s}^2 - \tau^2\right) - \frac{2\rho Q}{n}(\overline{s} - \tau) = 0.$$

At an internal solution, $\overline{s} > \tau$, so this expression becomes

$$\frac{1}{2}(\overline{s} + \tau) - \frac{2\rho Q}{n} = 0.$$

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(\overline{s} + R + \frac{\rho Q}{n}\right) - \frac{2\rho Q}{n} = 0 \quad \Longrightarrow \quad \frac{3\rho Q}{n} = \overline{s} + R.$$

Solving the optimality condition associated with $R^\star$ gives (removing the constant n/ρ)

$$\frac{1}{2}\left(\tau^2 - \underline{s}^2\right) - 2R(\tau - \underline{s}) = 0.$$

At an internal solution, $\underline{s} < \tau$, so this expression becomes

$$\frac{1}{2}(\tau + \underline{s}) - 2R = 0.$$

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(R + \frac{\rho Q}{n} - \underline{s}\right) - 2R = 0 \quad \Longrightarrow \quad 3R = \underline{s} + \frac{\rho Q}{n}.$$

Together these equations yield the linear system

$$\begin{aligned}\frac{3\rho Q}{n} =& \overline{s} + R, \\ 3R =& \underline{s} + \frac{\rho Q}{n}.\end{aligned}$$

It is straightforward to see that the solution is

$$Q^\star = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n, \quad R^\star = \frac{\overline{s} + 3\underline{s}}{8}.$$

The signal transition threshold at the optimum is $\tau(Q^\star, R^\star) = (\overline{s} + 3\underline{s})/8 + (3\overline{s} + \underline{s})/8 = (\overline{s} + \underline{s})/2$; then at the optimum both the maximum quantity and the reserve price are binding, as assumed.

The standard monopoly problems are straightforward. The quantity-monopoly problem is

$$\max_Q \mathbb{E}_s \left[Qv\left(\frac{Q}{n}; s\right)\right] = \max_Q Qv\left(\frac{Q}{n}; \mathbb{E}_s\left[s\right]\right) = \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - \rho Q\right) Q.$$

Then optimal quantity is $Q^M = (\overline{s} + \underline{s})/(4\rho)$. The price-monopoly problem is

$$\max_R \mathbb{E}_s \left[nR\varphi\left(R; s\right)\right] \propto \max_R R\varphi\left(R; \mathbb{E}_s\left[s\right]\right) \propto \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - R\right) R.$$

Then optimal price is $R^M = (\overline{s} + \underline{s})/4$.