Efficient Trade with Interdependent Values

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Abstract

This paper studies the possibility of efficiently trading a single divisible good among agents with interdependent values. We characterize the set of interdependence coefficients for which it is possible to design an efficient trading mechanism, and we analyze the possibility of efficiency in large markets. We find that efficiency is always achievable as long as the market is sufficiently large, although the level of efficiency does not necessarily increase monotonically in market size.

1 Introduction

Consider the problem of trading a divisible good among n agents with interdependent valuations. Can efficient trading mechanisms exist in such a market if each agents valuation of the good is dependent on all agents private information? Moreover, can such mechanisms, if they exist, satisfy Bayesian incentive compatibility (IC), interim individual rationality (IR) and ex-post budget balance (BB)?

In a seminal work, Myerson and Satterthwaite (1983) focus on a special bilateral trade environment, where each agents valuation is assumed to depend only on his own information. Their main result is that the difficulty caused by private information is insurmountable and no efficient mechanisms can satisfy IC, IR and BB at the same time. In a partnership dissolution

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setting, Cramton et al. (1987, henceforth CGK) generalize Myerson and Satterthwaites model by allowing the presence of multiple agents and a joint ownership structure. In contrast to Myerson and Satterthwaite, CGK find that efficiency might be possible if the initial property allocation is close to symmetry. Particularly, CGK show that the existence of efficient dissolution mechanisms is guaranteed as long as all agents are ex-ante identical.

We study a canonical model of interdependent values in which each agent's value is a weighted average of their own signal and other agents' signals, and the agents are ex-ante symmetric. We refer to the weight put on the mean of all agents' signals as the coefficient of interdependence; this coefficient converges to infinity as we approach the common values, and it is zero in the independent values case. We allow an arbitrary initial distribution of holdings in the traded good, and we do not require the agents' to be symmetric in this regard. This setting incorporates both Myerson and Satterthwaites and CGKs models as special cases. It also allows agents valuations to be interdependent, a phenomenon that naturally arises in many cases. For instance, in a stock market, investors may have correlated values of the publicly traded company, as one can only sell his shares at a price for which others are willing to pay, and thus his own valuation of the company depends on the others signals. In a partnership, it is reasonable that each partner has his own information about the joint venture and his own valuation is positively correlated to those of the others. For instance, consider Sony-Ericsson, a joint venture between Sony Corporation, a consumer and professional electronics company, and Ericsson, a Swedish provider of communication technology and services. Because Sony and Ericsson focus on different areas of the joint venture, namely electronics and communication respectively, it is natural that each party would have its own private information and that their valuations are interdependent. Eventually, Sony bought Ericsson's fifty percent stake in the joint venture, which provides a good example of the trading problem analyzed in our paper.

We first derive the necessary and sufficient condition for the existence of efficient trading mechanisms, we show that the set of interdependence coefficients for which efficient mechanisms exist is a (possibly empty or degenerate) interval, and we identify its bounds. The main forerunner of this result is Fieseler et al. (2003, henceforth FKM) who study partnership dissolution with interdependent values and provide the necessary and sufficient condition for the existence of an ex-ante budget balanced, Bayesian incentive-compatible, interim individually rational, and efficient mechanism. FKM show that efficiency is more difficult to achieve when agents (positive) valuation interdependence exists than when it does not, but they do not analyze the set of interdependence coefficients for which efficient trading mechanisms exist.

Our second main result shows that efficient mechanisms exist if there are sufficiently many agents.[1] This result is related to the common intuition that large markets facilitate efficient trade. Assuming private values, Gresik and Satterthwaite (1983) show that the second best mechanisms converge to ex post efficiency, and Wilson (1985) shows that a $k-$double auction is interim incentive efficient (that is not interim Pareto-dominated by another incentive compatible mechanism). In the context of interdependent values, the prior literature focused on double auctions and other selected market mechanisms, for instance, Reny and Perry (2006), Cripps and Swinkels (2006) and Fudenberg, Mobius and Seidel (2007) focus on double auctions and demonstrate approximate efficiency in large markets. An exception is Gresik (1991) who, in a simulation, analyzes the problem of trading N homogeneous goods among N buyers and N sellers, where each seller initially owns one good and each buyer can acquire at most one good as well. Assuming agents' types are drawn independently from the *uniform(0,1)* distribution, Gresik empirically shows that the level of efficiency increases as market size grows.[2]

Furthermore, our large market result implies that the second-best mechanisms are in fact first-best provided the market is sufficiently large. This is a new insight even though second-best mechanisms have been studied

[1] In particular, the efficiency means that the traders' information is aggregated in large markets.

[2] Gresik's model is different from ours in that he requires that one agent can have at most one unit share of the entire market, even if this agent has a valuation that is strictly higher than anyone else's. In contrast, we allow an agent to own the entire market share if his valuation is the highest. Also, we provide an analytical analysis while he provides a simulation.

in small markets. Jehiel and Pauzner (2006) and Loertscher and Wasser (2015), among others, analyze mechanisms that are second-best in the sense that they maximize the social welfare subject to incentive, participation, and budget constraints. Jehiel and Pauzner analyze a partnership dissolution problem, where the partnership is formed between two agents. They further require that both agents always have the same valuation of the partnership, and only one of them is informed in the sense that he can know the common valuation. Loertscher and Wasser relax Jehiel and Pauzners assumption of two-agent partnership, allow all parties to be informed, and weaken the assumption of common value. In this more general setting, Loertscher and Wasser derive the optimal mechanism that maximizes any convex combination of social welfare and designers revenue. In particular, they provide a characterization of the second-best mechanism by imposing budget balancedness.

One might expect that as the market grows, the set of interdependence coefficients that allow for efficient trade grows monotonically as well. Our last point is to show that this natural conjecture of monotonicity is false: we provide a counterexample illustrating this point in Section 3.

The rest of this paper is organized as follows: In Section 2, we describe the theoretical framework and give several definitions that are important for the subsequent analysis. In Section 3, we give the equivalence condition for the existence of efficient mechanisms, and we derive our result on large markets. We also give two examples in this section to illustrate how market size affects efficiency. Section 4 contains concluding remarks. The proof for the existence condition for efficient trade is provided in the Appendix.

2 Model

Consider a market formed by $n \geq 2$ agents with interdependent valuations for a single divisible good that is being traded. Each agent $i \in I = \{1, 2, ..., n\}$ has a privately known type $v_i \in [\underline{v}, \bar{v}] \subset R$. It is common knowledge that all signals v_i are drawn independently from the same distribution F, where the support is $[\underline{v}, \bar{v}]$ and the density function f is continuous and positive. We assume that F can be extended to $\tilde{F}$ such that $\tilde{F}(v) = F(v)$ for $v \in [\underline{v}, \bar{v}]$ and $\tilde{F}$ is real-analytical for some open interval containing $\bar{v}$. Let $r_i \in [0,1]$ be the share of the good agent i initially owns and define $\overrightarrow{r} := (r_1, r_2, ..., r_n)$, then a ***market*** is defined as $(n, \overrightarrow{r}, F)$.

Agent i's valuation of the good being traded is

$$v_i + \frac{\beta}{n}\sum_{j=1}^{n} v_j,$$

where β represents the degree to which her valuation is interdependent with others' valuations. In other words, β measures the strength of informational externalities associated with the market $(n, \overrightarrow{r}, F)$. In this paper, we assume β is non-negative for any market we consider.

A mechanism determines how many shares each participant gets and how much money she should pay for it. We study only incentive compatible and individually rational direct mechanisms characterized by the outcome functions $\{s_i(\overrightarrow{v}),\ t_i(\overrightarrow{v})\}_{i \in I}$, where, given a reported profile of types $\overrightarrow{v}$, $s_i(\overrightarrow{v})$ is the ex-post share of the good and $t_i(\overrightarrow{v})$ is the ex-post net money transfer agent i receives. This is without loss of generality by the Revelation Principle. We require that the share allocations must balance, that is, $\sum_{i=1}^{n} s_i(\overrightarrow{v}) = 1$. Moreover, ***ex-post budget balance (BB)***, $\sum_{i=1}^{n} t_i(\overrightarrow{v}) = 0$, must hold. Note that here the total share of the divisible good is normalized to unity so $s_i(\overrightarrow{v}) \in [0,1]$ always holds.

We restrict our attention to the case where each agent's utility is linear in the share and the net money transfer she receives, so given a reported type profile $\overrightarrow{v} =< v_i, v_{-i} >$, the ***ex-post utility*** of agent i is defined as

$$u_i(v_i, v_{-i}) = t_i(v_i, v_{-i}) + s_i(v_i, v_{-i}) * (v_i + \frac{\beta}{n}\sum_{j=1}^{n} v_j).$$

Let $E_{-i}[\ ]$ be the expectation operator with respect to v_{-i}, then the ***interim expected utility*** of agent i is

$$U_i(v_i) = T_i(v_i) + (1 + \frac{\beta}{n})v_i S_i(v_i) + E_{-i}[\frac{\beta}{n}\sum_{j \neq i} v_j s_i(v_i, v_{-i})],$$

where $T_i(v_i) = E_{-i}[t_i(v_i, v_{-i})]$ and $S_i(v_i) = E_{-i}[s_i(v_i, v_{-i})]$ are the interim expected net money transfer and allocation i receives respectively.

A mechanism is interim incentive compatible if truthful reporting is a dominant strategy for each type of agent; therefore here in our model, ***(interim) incentive compatibility(IC)*** just means

$$T_i(v_i) + (1 + \frac{\beta}{n})v_i S_i(v_i) + E_{-i}[\frac{\beta}{n}\sum_{j \neq i} v_j s_i(v_i, \overrightarrow{v_{-i}})]$$

$$\geq T_i(w) + (1 + \frac{\beta}{n})v_i S_i(w) + E_{-i}[\frac{\beta}{n}\sum_{j \neq i} v_j s_i(w, \overrightarrow{v_{-i}})]$$

for all $i \in I$, and $v_i, w \in [\underline{v}, \bar{v}]$. In other words, incentive compatibility is satisfied if and only if no agent can gain from misreporting her type.

A mechanism $\{s_i(\overrightarrow{v}),\ t_i(\overrightarrow{v})\}_{i \in I}$ is ***(interim) individually rational(IR)*** if and only if

$$T_i(v_i) + (1 + \frac{\beta}{n})v_i S_i(v_i) + E_{-i}[\frac{\beta}{n}\sum_{j \neq i} v_j s_i(v_i, \overrightarrow{v_{-i}})]$$

$$\geq r_i[v_i + \frac{\beta}{n}v_i + \frac{\beta(n-1)}{n}\bar{v}_0]$$

holds for all $i \in I$ and $v_i \in [\underline{v}, \bar{v}]$, where $\bar{v}_0 = E_i[v_i]\ \forall i$ is the expectation of each agent's type. This requirement can be interpreted as each agent being more willing than not to participate in the mechanism.

3 Main Results

We first derive the equivalence condition for the existence of incentive compatible (IC) and individually rational (IR) mechanisms that satisfy budget balance (BB). For economy of expression, for any market $(n, \overrightarrow{r}, F)$ and any share allocation rule $\{s_i(\overrightarrow{v})\}_{i \in I}$, we define v_i^* as any $v_i \in [\underline{v}, \bar{v}]$ such that $S_i(v_i) = r_i$. Moreover, in this paper, we focus on mechanisms such that $S_i(v_i)$ is continuous with r_i in its range, which is satisfied by efficient mechanisms.

Lemma 1. *In a market* $(n, \overrightarrow{r}, F)$ *with the interdependence coefficient* β*, for any allocation rules* $\{s_i(\overrightarrow{v})\}_{i\in I}$ *such that* S_i *is weakly increasing for each* $i \in I$*, there exists a transfer function* t *such that the mechanism* $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ *is incentive compatible and individually rational if and only if*

$$(1+\frac{\beta}{n})\sum_{i=1}^{n}\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdS_i(p) - \int_{\underline{v}}^{v_i^*} F(p)pdS_i(p)\} \tag{L1-1}$$

$$+\frac{\beta}{n}\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i} E_{-i}[v_j s_i(v_i, v_{-i})]dF(v_i) - \frac{\beta(n-1)}{n}\bar{v}_0 \geq 0$$

for all $i \in I$*.*

Lemma 1 gives a necessary and sufficient condition for the existence of incentive compatible, individually rational, and budget-balanced mechanisms. The *Only If* part is rather straightforward and follows from Lemmas 1-3 (see Appendix), which are derived directly using the definitions of IC and IR, while the *If* part is established by constructing an ex-post budget-balanced transfer function. Since the proof of this part is relatively tedious, it is provided in the Appendix. It should be noted that FKM provides a different strategy for this equivalence condition. The transfer function we construct satisfies ex-post budget balance while FKM's is only ex-ante budget-balanced.

Among all mechanisms satisfying IC, IR, and BB, we are interested in the ones that are Pareto efficient. In other words, assuming truth-telling, we are looking for mechanisms that maximize the social welfare

$$W := E_{All}\{\sum_{i=1}^{n}[s_i(v_i, \overrightarrow{v_{-i}}) * (v_i + \frac{\beta}{n}\sum_{i=1}^{n} v_i)]\},$$

where $E_{All}[\ ]$ is the expectation operator with respect to $\overrightarrow{v}$. Therefore, for a mechanism to be efficient, given a profile of reported types $\overrightarrow{v}$, the mechanism designer should compare each agent's valuation of the asset, $\{v_i + \frac{\beta}{n}\sum_{j=1}^{n} v_j\}_{i\in I}$, and assign some agent the entire share if and only if her valuation is the highest. Obviously this is equivalent to assigning the entire share to the agent with the highest type.

Definition 1. *A mechanism* $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ *is* ***ex-post (Pareto) efficient*** *if*

$$s_i(v_i, \overrightarrow{v_{-i}}) = \begin{cases} 0 & if\ v_i < max\{v_i, \overrightarrow{v_{-i}}\} \\ 1 & if\ v_i = max\{v_i, \overrightarrow{v_{-i}}\}. \end{cases} \tag{D1}$$

Note that the efficient mechanism satisfies our assumption that $S_i(v_i)$ is continuous, weakly increasing and has r_i in its range, so all the results we have so far can be applied. Therefore, we can plug this definition into inequality (L1-1) to derive Theorem 1, which answers the question: When is it possible to find an efficient mechanism such that it satisfies IC, IR and BB? For economy of expressions, henceforth we call such mechanisms *efficient trading mechanisms*, and we say that the good can be *traded efficiently* if such mechanisms exist.

Theorem 1. *In a market* $(n, \overrightarrow{r}, F)$ *with the interdependence coefficient* β*, the good can be traded efficiently if and only if*

$$(1+\frac{\beta}{n})\sum_{i=1}^{n}\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)\} \tag{T1-1}$$

$$+\frac{\beta}{n}\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}[F(v_i)^{n-2}(\int_{\underline{v}}^{v_i} v_j f(v_j)dv_j)]dF(v_i) - \frac{\beta(n-1)}{n}\bar{v}_0 \geq 0,$$

where $v_i^* = F^{-1}[(r_i)^{\frac{1}{n-1}}]$ *and* $G(v_i) = F(v_i)^{n-1}$.

Proof. Under the efficient allocation rule

$$s_i(v_i, \overrightarrow{v_{-i}}) = \begin{cases} 0 & if\ v_i < max\{v_i, \overrightarrow{v_{-i}}\} \\ 1 & if\ v_i = max\{v_i, \overrightarrow{v_{-i}}\}, \end{cases} \tag{D1}$$

we have by independence that

$S_i(v_i) = Pr\{v_i > max_{j\neq i} v_j\} = F(v_i)^{n-1} = G(v_i)$, and thus $v_i^* = F^{-1}[(r_i)^{\frac{1}{n-1}}]$.

Plug (D1) into inequality (L1-1) and we can get

$$(1+\frac{\beta}{n})\sum_{i=1}^{n}\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdG(p)-\int_{\underline{v}}^{v_i^*}F(p)pdG(p)\} \tag{T1-2}$$

$$+\frac{\beta}{n}\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}Pr\{v_i>max_{j\neq i}v_j\}E_{-i}[v_j|v_j\leq v_i]\}dF(v_i)-\frac{\beta(n-1)}{n}\bar{v}_0\geq 0.$$

Note that $E_{-i}[v_j|v_j\leq v_i]=\frac{\int_{\underline{v}}^{v_i}v_jf(v_j)dv_j}{F(v_i)}$, so (T1-2) simplies to (T1-1). □

A natural question is what the set of β is for which efficient mechanisms can exist. The following Theorem 2 allows us to fully characterize this set of β.

To shorten the expressions, for any fixed n, $\overrightarrow{r}$ and F, we define the following notations from Theorem 1:

$$T_1:=\sum_{i=1}^{n}\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdG(p)-\int_{\underline{v}}^{v_i^*}F(p)pdG(p)\},$$

$$T_2:=\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}[F(v_i)^{n-2}(\int_{\underline{v}}^{v_i}v_jf(v_j)dv_j)]dF(v_i)-(n-1)\bar{v}_0,$$

and

$$\beta^*:=\frac{-nT_1}{T_1+T_2}.$$

Note that Theorem 1 implies that β^* is the cutoff valuation interdependence coefficient for which efficiency is achievable.

Theorem 2. *In a market $(n,\overrightarrow{r},F)$ with the interdependence coefficient β, if $\beta^*<0$, there do no exist efficient mechanisms. Otherwise the good can be traded efficiently if and only if $\beta\in[0,\beta^*]$.*

Proof. Note that inequality (T1-1) can be simplified to $(1+\frac{\beta}{n})T_1+\frac{\beta}{n}T_2\geq 0$, which is equivalent to $nT_1+\beta[T_1+T_2]\geq 0$. Therefore to prove Theorem 2, we only need to show $T_1+T_2<0$ holds. Note that T_1 is concave in $\overrightarrow{r}$ and T_2 is not affected by $\overrightarrow{r}$, so it suffices to consider the equal-share initial allocation only.

By definition,

$$T_1 + T_2 \tag{T2-1}$$

$$= \sum_{i=1}^{n} \{ \int_{v_i^*}^{\bar{v}} [1 - F(p)] p dG(p) - \int_{\underline{v}}^{v_i^*} F(p) p dG(p) \}$$

$$+ n(n-1) \int_{\underline{v}}^{\bar{v}} [F(v_i)^{n-2} (\int_{\underline{v}}^{v_i} v_j f(v_j) dv_j)] dF(v_i) - (n-1)\bar{v}_0$$

for any $i \neq j$.

First note that integration by parts implies equalities (T2-2) to (T2-4) below.

$$\int_{v_i^*}^{\bar{v}} p dG(p) = \bar{v} - v_i^* F(v_i^*)^{n-1} - \int_{v_i^*}^{\bar{v}} G(p) dp \tag{T2-2}$$

$$\int_{\underline{v}}^{\bar{v}} F(p) p dG(p) = \frac{n-1}{n} [\bar{v} - \int_{\underline{v}}^{\bar{v}} F^n(p) dp], \tag{T2-3}$$

$$\int_{\underline{v}}^{\bar{v}} [F(v_i)^{n-2} (\int_{\underline{v}}^{v_i} v_j f(v_j) dv_j)] dF(v_i) \tag{T2-4}$$

$$= \frac{\bar{v}_0}{n-1} - \frac{\bar{v}}{n(n-1)} + \frac{1}{n(n-1)} \int_{\underline{v}}^{\bar{v}} F(v_i)^n dv_i,$$

where (T2-3) uses the fact that

$$\int_{\underline{v}}^{\bar{v}} p dF^n(p) = \frac{n}{n-1} \int_{\underline{v}}^{\bar{v}} p F(p) dF^{n-1}(p) = \frac{n}{n-1} \int_{\underline{v}}^{\bar{v}} F(p) p dG(p).$$

With equalities (T2-1) and (T2-2), we have

$$\int_{v_i^*}^{\bar{v}} [1 - F(p)] p dG(p) - \int_{\underline{v}}^{v_i^*} F(p) p dG(p) \tag{T2-5}$$

$$= \frac{\bar{v}}{n} - v_i^* F(v_i^*)^{n-1} - (\int_{v_i^*}^{\bar{v}} F^{n-1}(p) dp - \int_{\underline{v}}^{\bar{v}} F^n(p) dp) - \frac{1}{n} \int_{\underline{v}}^{\bar{v}} F^n(p) dp.$$

Plug equations (T2-4) and (T2-5) back into (T2-1) and apply integration by parts again to get

$$T_1 + T_2 \tag{T2-6}$$

$$= \sum_{i=1}^{n} \int_{\underline{v}}^{v_i^*} [F^n(p) - \frac{1}{n}F(p)]dp + \sum_{i=1}^{n} [\frac{1}{n}\bar{v} - \bar{v}F(v_i^*)^{n-1}]$$

$$- \sum_{i=1}^{n} \int_{v_i^*}^{\bar{v}} \{[F^{n-1}(p) - F^n(p)] + \frac{1}{n}F(p) - F(v_i^*)^{n-1}\}dp$$

after some tedious algebra.

Note that the first term (after the equality sign) in (T2-6) is strictly negative since $x^n - \frac{x}{n} < 0$ for all $x \in (0, (\frac{1}{n})^{\frac{1}{n-1}})$, and the second term is zero as we have $F(v_i^*)^{n-1} = S_i(v_i^*) = r_i$ for all i under the efficient allocation rule, so it suffices to show that $\sum_{i=1}^{n} \int_{v_i^*}^{\bar{v}} \{[F^{n-1}(p) - F^n(p)] + \frac{1}{n}F(p) - F(v_i^*)^{n-1}\}dp > 0$, but this is just because

$$nx^{n-1} - nx^n + x - 1 = (x-1)(1 - nx^{n-1})$$

and both $x - 1 < 0$ and $1 - nx^{n-1} < 0$ hold for $x \in ((\frac{1}{n})^{\frac{1}{n-1}}, 1)$.

Therefore $T_1 + T_2 < 0$ holds. □

Theorem 2 characterizes the set of β for which efficient mechanisms exist as an interval (if the asset can be traded efficiently), including the case where the interval is degenerate.

Given a market $(n, \overrightarrow{r}, F)$, Theorem 2 says that efficient trading mechanisms might not exist if the interdependence coefficient β is too large. On the other hand, we show that no matter what the initial allocation $\overrightarrow{r}$ is, efficiency can always be achieved if we replicate the economy for sufficiently many times. This idea is formalized in Theorem 3 below. Before we move on to Theorem 3, let's first recall the definition of a replication of an economy.

Definition 2. *For the market* $(n, \overrightarrow{r}, F)$ *where* $\overrightarrow{r} = (r_1, r_2, ..., r_n)$, *its* ***m-fold replication*** *is the N-agent economy* $(N, \overrightarrow{r}, F)$, *where N=mn,* $\overrightarrow{r} = (r_1, r_2, ..., r_i, ..., r_N)$ *and*

$$r_i = \frac{r_k}{m}$$

for $i = (k-1)m + 1, (k-1)m + 2, ..., km$; $k = 1, 2..., n$.

For example, suppose a market is $(n, \overrightarrow{r}, F)$ where the initial ownerships are $\overrightarrow{r} = (1, 0, ..., 0)$, then its m-fold replication is $(mn, \overrightarrow{r}, F)$, *where* $\overrightarrow{r} =$

$(r_i = \frac{1}{m}$ *for* $i = 1, 2, ..., m;\ r_i = 0$ *for* $i = m+1, m+2, ..., mn)$.

Next we formalize our results on large markets in Theorem 3. The proof utilizes the two lemmas below, whose proofs are saved for the Appendix.

Lemma 2. *The equality*

$$T_2 = \bar{v}_0 - E[max\{v_1, v_2, ..., v_n\}] \tag{L2}$$

holds for any market $(n, \overrightarrow{r}, F)$.

The general idea of the proof for Theorem 3 is to represent every term involving the type distribution F in a way that is easy to manipulate, since F is allowed to be completely arbitrary and thus is troublesome. Lemma 2 deals with T_2, the second part in the equivalence condition for efficiency, and it proves that T_2 will eventually stabilize to a negative constant as market size grows. This part is to be used with part (b) of Lemma 3, and its proof is relatively easy. The key step in deriving Theorem 3 is to utilize part (a) of Lemma 3, which approximates F near the boundary of its domain. While F is completely arbitrary, this approximation turns it into functions that are nice to work with, and this is essential for us to establish our result on large markets. Lemma 3 itself is proved using Taylor Approximation along with the Dominance Convergence Theorem.

Lemma 3. *For the market* $(N, \overrightarrow{r}, F)$, *where* $N = mn$ *and* $\overrightarrow{r} = (r_i = \frac{1}{m}$ *for* $i = 1, 2, ..., m;\ r_i = 0$ *for* $i = m+1, m+2, ..., N)$, *if* m *is sufficiently large, then*

(a) there exist some positive integer α *and some positive real numbers* A_1, A_2 *such that*

$$\bar{v} - v_i^* \approx A_1 \left(\frac{logm}{N-1}\right)^{\frac{1}{\alpha}} \tag{L3-1}$$

for $i = 1, 2, ..., m$, *and*

$$\int_{\underline{v}}^{\bar{v}} F^N(p)dp \approx A_2 N^{-\frac{1}{\alpha}}; \tag{L3-2}$$

(b) T_1 *is bounded.*

Theorem 3. *For a market $(n, \overrightarrow{r}, F)$ with an initial ownership structure $\overrightarrow{r} = (r_1, r_2, ..., r_n)$, consider its m-fold replication $(mn, \overrightarrow{r}, F)$. As m goes to infinity, the cutoff β^* associated with its m-fold replication also goes to positive infinity.*

Proof. Note the replicated market has the size of $N = mn$. By Lemma 2, $T_2 = \bar{v}_0 - E[max_i\{v_1, v_2, ..., v_N\}]$. It can be shown that $E[max_i\{v_1, v_2, ..., v_N\}]$ goes to $\bar{v}$ as N goes to infinity, therefore $T_1 + T_2$ is bounded by Lemma 3 (b), and thus we only need to prove that NT_1 goes to positive infinity.

By the definition of T_1,

$$NT_1 = N\sum_{i=1}^{N}\{\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)\}.$$

For $i = 1, 2, ..., m$, by equation (T2-5), we have

$$\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p) \tag{T3-1}$$

$$= \frac{\bar{v}}{N} - \frac{nv_1^*}{N} - (\int_{v_1^*}^{\bar{v}} F^{N-1}(p)dp - \int_{\underline{v}}^{\bar{v}} F^N(p)dp) - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^N(p)dp,$$

where v_1^* is without loss of generality since the allocations are the same. Note that

$$\int_{v_1^*}^{\bar{v}} F^{N-1}(p)dp - \int_{\underline{v}}^{\bar{v}} F^N(p)dp$$
$$\leq \int_{v_1^*}^{\bar{v}}[F^{N-1}(p) - F^N(p)]dp$$
$$\leq (\bar{v} - v_1^*)max_{x\in[0,1]}\{x^{N-1} - x^N\}$$
$$= (\bar{v} - v_1^*)(\frac{N-1}{N})^{N-1}\frac{1}{N}.$$

Also, when N approaches infinity, $(\frac{N-1}{N})^{N-1}\frac{1}{N} \approx \frac{1}{e}\frac{1}{N}$. Therefore for $i = 1, 2, ..., m$, from equation (T3-1) we have

$$\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p) \tag{T3-2}$$

$$\geq \frac{\bar{v}-v_1^*}{N}(n-\frac{1}{e}) - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^N(p)dp - \frac{n-1}{N}\bar{v}$$

as N approaches infinity. On the other hand, for $i = m+1, m+2, ..., mn$, we have

$$\int_{v_i^*}^{\bar{v}}[1-F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p) \qquad \text{(T3-3)}$$

$$\geq \frac{\bar{v}}{N} - (\int_{\underline{v}}^{\bar{v}} F^{N-1}(p)dp - \int_{\underline{v}}^{\bar{v}} F^N(p)dp) - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^N(p)dp$$

$$\approx \frac{\bar{v}}{N} - A_2[(N-1)^{\frac{-1}{\alpha}} - N^{\frac{-1}{\alpha}}] - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^N(p)dp,$$

where the approximation follows from Lemma 3 (a) as N approaches infinity.

Combine (T3-2) and (T3-3) and apply Lemma 3 (a), then the following inequality holds when N goes to infinity.

$$NT_1 \geq \frac{N}{N^{\frac{1}{\alpha}}}\{A_1(log\frac{N}{n})^{\frac{1}{\alpha}}(1-\frac{1}{en}) - A_2 - (1-\frac{1}{n})A_2 N^{1+\frac{1}{\alpha}}[(N-1)^{\frac{-1}{\alpha}} - N^{\frac{-1}{\alpha}}]\}.$$

Note that $N^{1+\frac{1}{\alpha}}[(N-1)^{\frac{-1}{\alpha}} - N^{\frac{-1}{\alpha}}]$ goes to a constant, therefore NT_1 goes to infinity and we have completed the proof. $\square$

For any arbitrary market $(n, \overrightarrow{r}, F)$, Theorem 3 asserts the existence of efficient trading mechanisms as long as we replicate the economy for sufficiently many times, no matter what the parameters of the market are. This idea coincides with the classical Core-Convergence Theorem, which states that approximate competitiveness can be achieved as long as the size of an economy is large enough. It should be noted that Theorem 3 only characterizes the behavior of the cutoff β^* in the limit. One might conjecture that β^* is monotonically increasing in n, since trade can be efficient as long as we have enough agents. As the following example shows, however, this conjecture is not necessarily correct.

Example 1. *Depending on the market $(n, \overrightarrow{r}, F)$, the cutoff interdependence coefficient, $\beta^* := \frac{-nT_1}{T_1+T_2}$, might or might not increase monotonically in market size.*

C:/Users/LarryP/Desktop/UCLA/Spring 2015/Thesis/Matlab Files/11132125.eps

Figure 1: Plot of β^*

We will give two examples of markets, one in which β^* increases monotonically in n and the other not.

Let the first market be such that the type distribution has the PDF of $f(x) := 0.1 + 0.1sin(x)$ with the range of $[0, 8.4438]$. Let the second market have the type distribution *uniform (0,1)*. Suppose in both markets the initial ownership structure is symmetric, that is, $r = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$. Then we plot the cutoff interdependence coefficient, β^*, against the market size, n, in Figure 1, where the blue line corresponds to the first market and red to the second. Obviously β^* is not necessarily monotonic in market size. $\square$

4 Conclusion

In this paper, we study the problem of trading a single divisible good across n agents, allowing the initial allocation of the good to be completely arbitrary and agents' valuations to be interdependent. For any market $(n, \overrightarrow{r}, F)$, we first derive the equivalence condition for the existence of an ex-post efficient mechanism that satisfies incentive compatibility, individual rationality, and budget balance. Based on this equivalence condition, we fully characterize the set of interdependence coefficients for which efficiency is achievable. Then, we analyze the effect of market size on possibility of efficiency. We prove that, for any market, full efficiency must be achievable as long as the market size is large enough, although the effect of an increase in market size on efficiency is not monotonic as demonstrated by the two examples above.

Appendix

A.1 Proof of Lemma 1. The proof of Lemma 1 employs the following two lemmas, whose proofs are relatively standard and follow directly from the definitions of IC and IR.

Lemma 4. *A mechanism* $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ *is incentive compatible if and only if* $S_i(v_i)$ *is weakly increasing and*

$$T_i(v_i) - T_i(w)$$

$$= -(1+\frac{\beta}{n})\int_w^{v_i} p dS_i(p) - \frac{\beta}{n}E_{-i}\{\sum_{j\neq i} v_j[s_i(v_i, \overrightarrow{v_{-i}}) - s_i(w, \overrightarrow{v_{-i}})]\} \qquad \text{(L4)}$$

holds for all $i \in I$ *and* $v_i, w \in [\underline{v}, \bar{v}]$.

Proof. Only if.

By definition, incentive compatibility implies

$$U_i(v_i) - U_i(w) \geq (1+\frac{\beta}{n})(v_i - w)S_i(w).$$

Therefore U_i has a supporting hyperplane at w with slope $(1+\frac{\beta}{n})S_i(w) \geq 0$ and is convex. Note that $\frac{dU_i}{dv_i} = (1+\frac{\beta}{n})S_i$, and $(1+\frac{\beta}{n}) > 0$ by our model assumption, so S_i is increasing.

By the Fundamental Theorem of Calculus, $U_i(v_i) - U_i(w) = \int_w^{v_i}(1+\frac{\beta}{n})S_i(p)dp$, which simplifies to equation (1) by integration by parts.

If. Adding the identify

$$(1+\frac{\beta}{n})v_i[S_i(v_i) - S_i(w)] = (1+\frac{\beta}{n})[v_i\int_w^{v_i} dS_i(p)]$$

to (L4) results in

$$0 \leq (1+\frac{\beta}{n})\int_w^{v_i}(v_i - p)dS_i(p) = U_i(v_i) - U_i(w),$$

which means $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ is incentive compatible . $\square$

Lemma 5. *Suppose a mechanism* $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ *is incentive com-*

patible, then it is individual rational if and only if

$$T_i(v_i^*) + \frac{\beta}{n}E_{-i}[\sum_{j\neq i} v_j s_i(v_i^*, \overrightarrow{v_{-i}})] - \frac{\beta(n-1)}{n} r_i \bar{v}_0 \geqslant 0 \qquad \text{(L5)}$$

for all $i \in I$.

Proof.

First note that for any incentive compatible mechanism $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$, agent i minimizes her expected net utility $U_i(v_i) - r_i[v_i + \frac{\beta}{n}v_i + \frac{\beta(n-1)}{n}\bar{v}_0]$ by reporting v_i^*. This is because the derivative of trader i's expected net utility is $(1+\frac{\beta}{n})(S_i - r_i)$, which is a positive constant times $(S_i - r_i)$, and S_i is weakly increasing by Lemma 4.

Individual rationality means that all agents would want to participate rather than not in the mechanism. It suffices to consider only v_i^* since it minimizes each agent's interim expected utility. So

$$T_i(v_i^*) + (1+\frac{\beta}{n})v_i^* S_i(v_i^*) + E_{-i}[\frac{\beta}{n}\sum_{j\neq i} v_j s_i(v_i^*, \overrightarrow{v_{-i}})]$$

$$\geq (1+\frac{\beta}{n}) r_i v_i^* + \frac{\beta(n-1)}{n} r_i \bar{v}_0,$$

where $S_i(v_i^*) = r_i$ by the definition of v_i^*. This inequality simplifies to inequality (L5). □

Now we are ready to prove Lemma 1.

Proof. Only if.

By Lemma 4, incentive compatibility implies

$$\int_{\underline{v}}^{\bar{v}} T_i(v_i) dF(v_i)$$

$$= T_i(v_i^*) - (1+\frac{\beta}{n})\int_{\underline{v}}^{\bar{v}}\int_{v_i^*}^{v_i} p dS_i(p) dF(v_i) - \frac{\beta}{n}\int_{\underline{v}}^{\bar{v}} E_{-i}\{\sum_{j\neq i} v_j[s_i(v_i, \overrightarrow{v_{-i}}) - s_i(v_i^*, \overrightarrow{v_{-i}})]\} dF(v_i),$$

which simplifies to

$$E_i[T_i(v_i^*)]$$

$$= T_i(v_i^*) - (1+\frac{\beta}{n})\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdS_i(p) - \int_{\underline{v}}^{v_i^*}F(p)pdS_i(p)\}$$

$$-\frac{\beta}{n}\int_{\underline{v}}^{\bar{v}}E_{-i}\{\sum_{j\neq i}v_j[s_i(v_i,\overrightarrow{v_{-i}}) - s_i(v_i^*,\overrightarrow{v_{-i}})]\}dF(v_i)$$

by changing the order of integration.

Budget balance requires $\sum_{i=1}^n t_i(\overrightarrow{v}) = 0$ for any realization of types $\overrightarrow{v}$, so ex-ante we have $\sum_{i=1}^n E_i[T_i(v_i)] = E_N[\sum_{i=1}^n t_i(\overrightarrow{v})] = 0$.

Therefore,

$$\sum_{i=1}^n T_i(v_i^*)$$

$$= (1+\frac{\beta}{n})\sum_{i=1}^n\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdS_i(p) - \int_{\underline{v}}^{v_i^*}F(p)pdS_i(p)\}$$

$$+\frac{\beta}{n}\sum_{i=1}^n\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}E_{-i}[v_j s_i(v_i,\overrightarrow{v_{-i}})]dF(v_i) - \frac{\beta}{n}\sum_{i=1}^n\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}E_{-i}[v_j s_i(v_i^*,\overrightarrow{v_{-i}})]dF(v_i)$$

$$\geq \frac{\beta(n-1)}{n}\bar{v}_0 - \frac{\beta}{n}\sum_{i=1}^n\sum_{j\neq i}E_{-i}[v_j s_i(v_i^*,\overrightarrow{v_{-i}})],$$

where the inequality holds by Lemma 5, and this simplifies to the inequality desired.

Now we prove the second part.

If. In terms of notation, here we use v_i and $\hat{v}_i$ to represent agent i's true and reported types, respectively. In equilibrium they are the same.

First we define the ex-post transfer function

$$t_i(\hat{v}_i, v_{-i})$$

$$= c_i - (1+\frac{\beta}{n})\int_{\underline{v}}^{\hat{v}_i}pdS_i(p) + \frac{1}{n-1}(1+\frac{\beta}{n})\sum_{j\neq i}\int_{\underline{v}}^{v_j}pdS_j(p)$$

$$+\frac{\beta}{n}\int_{\underline{v}}^{\bar{v}}\sum_{j\neq i}[v_j s_i(p,\overrightarrow{v_{-i}}) - v_j s_i(\hat{v}_i,\overrightarrow{v_{-i}})]dF(p) + g_i(\hat{v}_i, v_{-i}),$$

where $g_i(v_i, v_{-i})$ is defined as

$$g_i(v_i, v_{-i}) := \frac{\beta}{\text{n(n-1)}} \sum_{j \neq i} \int_{\underline{v}}^{\bar{v}} \sum_{k \neq j} [v_k s_j(v_j, \overrightarrow{v_{-j}}) - v_k s_j(p, \overrightarrow{v_{-j}})] dF(p),$$

and c_i is a constant to be defined later.

Note that $g_i(v_i, v_{-i})$ is defined such that

$$\sum_{i=1}^{n} g_i(\hat{v}_i, v_{-i}) + \sum_{i=1}^{n} \frac{\beta}{n} \int_{\underline{v}}^{\bar{v}} \sum_{j \neq i} [v_j s_i(v_i, \overrightarrow{v_{-i}}) - v_j s_i(\hat{v}_i, \overrightarrow{v_{-i}})] dF(v_i) = 0.$$

Now for any given vector $< v_i, v_{-i} >$, budget balance holds if and only if $\sum_{i=1}^{n} t_i(v_i, v_{-i}) = 0$, that is,

$$\sum_{i=1}^{n} c_i - (1 + \frac{\beta}{n}) \sum_{i=1}^{n} \int_{\underline{v}}^{v_i} p dS_i(p) + \frac{1}{n-1}(1 + \frac{\beta}{n}) \sum_{i=1}^{n} \sum_{j \neq i} \int_{\underline{v}}^{v_j} p dS_j(p)$$

$$+ \frac{\beta}{n} \sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} \sum_{j \neq i} [v_j s_i(p, v_{-i}) - v_j s_i(v_i, v_{-i})] dF(p) + \sum_{i=1}^{n} g_i(v_i, \overrightarrow{v_{-i}}) = 0.$$

The second and the third terms in the above equation cancel each other by integration by parts, and the last two terms cancel each other by the definition of $g_i(v_i, v_{-i})$, therefore budget balance holds if and only if $\sum_{i=1}^{n} c_i = 0$. We will show this is true after introducing the definition of c_i. For now we continue to verify incentive compatibility and individual rationality.

After changing the order of integration, we get

$$T_i(\hat{v}_i) \tag{L1-2}$$

$$= c_i - (1 + \frac{\beta}{n}) \int_{\underline{v}}^{\hat{v}_i} p dS_i(p) + \frac{1}{n-1}(1 + \frac{\beta}{n}) \sum_{j \neq i} \int_{\underline{v}}^{\bar{v}} [1 - F(p)] p dS_j(p)$$

$$+ \frac{\beta}{n} E_{-i} \{ \int_{\underline{v}}^{\bar{v}} \{ \sum_{j \neq i} [v_j s_i(p, v_{-i}) - v_j s_i(\hat{v}_i, v_{-i})] dF(p) \} \} + E_{-i}[g_i(\hat{v}_i, v_{-i})]$$

Therefore

$$T_i(\hat{v}_i) - T_i(v_i^*)$$

$$= -(1+\frac{\beta}{n})\int_{v_i^*}^{\hat{v}_i} pdS_i(p) - \frac{\beta}{n}E_{-i}\{\sum_{j\neq i}[v_j s_i(\hat{v}_i, v_{-i}) - v_j s_i(v_i^*, v_{-i})]\}$$

$$+E_{-i}[g_i(\hat{v}_i, v_{-i})] - E_{-i}[g_i(v_i^*, v_{-i})],$$

where $E_{-i}[g_i(\hat{v}_i, v_{-i})] = 0 \ \forall \hat{v}_i$ follows from the definition of $g_i(v_i, v_{-i})$ and our model assumptions on integrability. Note that this is actually equation (L4) in Lemma 4 and thus $\{s_i(\overrightarrow{v}), t_i(\overrightarrow{v})\}_{i\in I}$ is incentive compatible.

Now for individual rationality, we have

$$\sum_{i=1}^{n} T_i(v_i^*) \tag{L1-3}$$

$$= \sum_{i=1}^{n} c_i + (1+\frac{\beta}{n})\sum_{i=1}^{n}\{\int_{v_i^*}^{\bar{v}}[1-F(p)]pdS_i(p) - \int_{\underline{v}}^{v_i^*} F(p)pdS_i(p)\}$$

$$+\frac{\beta}{n}\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}} E_{-i}\{\sum_{j\neq i}[v_j s_i(v_i, v_{-i}) - v_j s_i(v_i^*, v_{-i})]\}dF(v_i) + \sum_{i=1}^{n} E_{-i}[g_i(v_i^*, v_{-i})]$$

from (L1-2), so

$$\sum_{i=1}^{n} T_i(v_i^*) \geq \frac{\beta(n-1)}{n}\bar{v}_0 - \sum_{i=1}^{n}\frac{\beta}{n}E_{-i}[\sum_{j\neq i} v_j s_i(v_i^*, v_{-i})] + \sum_{i=1}^{n} E_{-i}[g_i(v_i^*, v_{-i})]$$

by inequality (L1-3) and the fact that $\sum_{i=1}^{n} c_i = 0$ (to be verified below).

Since $E_{-i}[g_i(v_i^*, v_{-i})] = 0$, this inequality simplies to $\sum_{i=1}^{n} T_i(v_i^*) \geq A$, where we define

$$A_i := r_i\frac{\beta(n-1)}{n}\bar{v}_0 - \frac{\beta}{n}E_{-i}[\sum_{j\neq i} v_j s_i(v_i^*, v_{-i})]$$

and

$$A := \sum_{i=1}^{n} A_i = \frac{\beta(n-1)}{n}\bar{v}_0 - \sum_{i=1}^{n}\frac{\beta}{n}E_{-i}[\sum_{j\neq i} v_j s_i(v_i^*, v_{-i})].$$

Now we can let

$$c_i = \frac{1}{n}\sum_{i=1}^{n}\{-(1+\frac{\beta}{n})\int_{\underline{v}}^{v_i^*} pdS_i(p) + \frac{1}{n-1}(1+\frac{\beta}{n})\sum_{j\neq i}\int_{\underline{v}}^{\bar{v}}[1-F(p)]pdS_j(p)$$

$$+\frac{\beta}{n}\int_{\underline{v}}^{\bar{v}} E_{-i}\{\sum_{j\neq i}[v_j s_i(v_i, v_{-i}) - v_j s_i(v_i^*, v_{-i})]\}dF(v_i) + E_{-i}[g_i(v_i^*, v_{-i})]\}$$

$$+(1+\frac{\beta}{n})\int_{\underline{v}}^{v_i^*} pdS_i(p) - \frac{1}{n-1}(1+\frac{\beta}{n})\sum_{j\neq i}\int_{\underline{v}}^{\bar{v}}[1-F(p)]pdS_j(p)$$

$$-\frac{\beta}{n}\int_{\underline{v}}^{\bar{v}} E_{-i}\sum_{j\neq i}[v_j s_i(v_i, v_{-i}) - v_j s_i(v_i^*, v_{-i})]dF(v_i) - E_{-i}[g_i(v_i^*, v_{-i})].$$

Obviously $\sum_{i=1}^{n} c_i = 0$ because the sum of all other terms cancel and thus the budget indeed balances. So by (L1-3) we actually have

$$c_i = \frac{1}{n}\sum_{i=1}^{n} T_i(v_i^*)$$

$$+(1+\frac{\beta}{n})\int_{\underline{v}}^{v_i^*} pdS_i(p) - \frac{1}{n-1}(1+\frac{\beta}{n})\sum_{j\neq i}\int_{\underline{v}}^{\bar{v}}[1-F(p)]pdS_j(p)$$

$$-\frac{\beta}{n}\int_{\underline{v}}^{\bar{v}} E_{-i}\sum_{j\neq i}[v_j s_i(v_i, v_{-i}) - v_j s_i(v_i^*, v_{-i})]dF(v_i) - E_{-i}[g_i(v_i^*, v_{-i})].$$

Plug $\hat{v}_i = v_i^*$ and c_i into equation (L1-2), then we get

$$T_i(v_i^*) = \frac{1}{n}\sum_{i=1}^{n} T_i(v_i^*) \geq \frac{1}{n}A.$$

However, this is not exactly what we want yet. Note that by Lemma 5, in order to prove individual rationality, what we need is to show $T_i(v_i^*) \geq A_i$, so here we still need some modifications to "shift" the definitions a little bit. One way to deal with it is to define $d_i := T_i(v_i^*) - A_i$, so we have $\sum_{i=1}^{n} d_i \geq 0$

by definition. Then we define

$$\hat{t}_i(\hat{v}_i, v_{-i}) := t_i(\hat{v}_i, v_{-i}) - d_i + \frac{\sum_{i=1}^n d_i}{n},$$

so budget balance still holds. Then

$$\hat{T}_i(v_i) := E_{-i}[t_i(v_i, v_{-i})] = T_i(v_i) - d_i + \frac{\sum_{i=1}^n d_i}{n},$$

so we can still show equation (L4) and thus incentive compatibility is no affected.

Finally,

$$\hat{T}_i(v_i^*) = T_i(v_i^*) - d_i + \frac{\sum_{i=1}^n d_i}{n} \geq A_i$$

since $\sum_{i=1}^n d_i \geq 0$. Therefore by Lemma 5, we have proved that$\{s_i(\overrightarrow{v}), \hat{t}_i(\overrightarrow{v})\}_{i\in I}$ is individually rational, and this completes the proof of Lemma 1. □

A.2 Proof of Lemma 2.

First recall that

$$T_2 := \sum_{i=1}^n \int_{\underline{v}}^{\bar{v}} \sum_{j\neq i} [F(v_i)^{n-2}(\int_{\underline{v}}^{v_i} v_j f(v_j) dv_j)] dF(v_i) - (n-1)\bar{v}_0.$$

Define $h(v_i) := \int_{\underline{v}}^{v_i} v_j f(v_j) dv_j$, then by integration by parts we have

$$\int_{\underline{v}}^{\bar{v}} \sum_{j\neq i} [F(v_i)^{n-2}(\int_{\underline{v}}^{v_i} v_j f(v_j) dv_j)] dF(v_i)$$

$$= (n-1)h(v_i)\frac{1}{n-1}F(v_i)^{n-1}|_{\underline{v}}^{\bar{v}} - (n-1)\int_{\underline{v}}^{\bar{v}} \frac{1}{n-1}F(v_i)^{n-1} v_i f(v_i) dv_i$$

$$= \bar{v}_0 - \int_{\underline{v}}^{\bar{v}} F(v_i)^{n-1} v_i f(v_i) dv_i$$

$$= \bar{v}_0 - \frac{1}{n}\int_{\underline{v}}^{\bar{v}} v_i dF(v_i)^n$$

$$= \bar{v}_0 - \frac{1}{n}E[max\{v_1, v_2, ..., v_n\}].$$

Plug the equality above into the definition of T_2 and the lemma is proved.

□

A.3 Proof of Lemma 3 part (a).

Equality L3-1. First we claim that $F(v) \approx 1-c(\bar{v}-v)^{\alpha}$ for some positive constant c and some positive integer α as v goes to $\bar{v}$. This will imply that $F(v)^N$ is approximately equal to $[1-c(\bar{v}-v)^{\alpha}]^N$ when N goes to infinity.

Using our assumption on real-analyticity of F, we can write the Taylor expansion

$$F(v) = F(\bar{v}) + F^{'}(\bar{v})(v-\bar{v}) + \frac{F^{''}(\bar{v})}{2}(v-\bar{v})^2 + ...$$

$$= 1 + f(\bar{v})(v-\bar{v}) + \frac{f^{'}(\bar{v})}{2}(v-\bar{v})^2 + ...$$

Now we claim that $F(v) \approx 1-c_{\alpha}(\bar{v}-v)^{\alpha}$ when v is very close to $\bar{v}$ for some $c_{\alpha} > 0$, which is the only case we care about as N goes to infinity.

Pick the smallest α (a positive integer) such that $F^{(\alpha)} \neq 0$. Such α must exist by analyticity.

Then

$$F(v) = 1 - c_{\alpha}(\bar{v}-v)^{\alpha} + c_{\alpha+1}(\bar{v}-v)^{\alpha+1} + ...$$

Since $F(v) < F(\bar{v})$ for $v < \bar{v}$, intuitively we would expect that $c_{\alpha} > 0$. To prove this, note that for small (as v is very close to $\bar{v}$) $\delta > 0$,

$$F(\bar{v}-\delta) = 1 - c_{\alpha}\delta^{\alpha} + c_{\alpha+1}\delta^{\alpha+1} + ...$$

Define $h(\delta) = c_{\alpha+1}\delta^{\alpha+1} + ...$, so $lim_{\delta\to 0}\frac{h(\delta)}{\delta^{\alpha+1}} = c_{\alpha+1}$. Therefore we can choose δ small enough such that $|\frac{h(\delta)}{\delta^{\alpha+1}}| < 2|c_{\alpha+1}|$, so $|h(\delta)| < 2\delta^{\alpha+1}|c_{\alpha+1}|$.

Note that $F(\bar{v}-\delta) < 1$, so $c_{\alpha}\delta^{\alpha} > h(\delta) > -2\delta^{\alpha+1}|c_{\alpha+1}|$, so $c_{\alpha} > -2\delta|c_{\alpha+1}|$. Since $-2\delta|c_{\alpha+1}|$ goes to 0 as δ goes to 0, our intuition before is indeed correct and $c_{\alpha} > 0$ must hold. Since all other terms are dominated when N is large enough, we have $F(v) \approx 1-c_{\alpha}(\bar{v}-v)^{\alpha}$ for $c_{\alpha} > 0$.

For simplicity of notation, we write $F(v) \approx 1-c_{\alpha}(\bar{v}-v)^{\alpha}$ as $F(v) \approx$

$1-c(\bar{v}-v)^{\alpha}$ and plug in $v=v_i^*$, where $i=1,2,...,m$. This gives

$$\bar{v}-v_i^* \approx (\frac{1-(\frac{1}{m})^{\frac{1}{N-1}}}{c})^{\frac{1}{\alpha}}.$$

Note that $\frac{1}{m}=e^{-logm}$, so

$$1-(\frac{1}{m})^{\frac{1}{N-1}} = 1-e^{\frac{-logm}{N-1}}.$$

Then by the Taylor expansion of e^x at $x=0$, we have

$$1-(\frac{1}{m})^{\frac{1}{N-1}} \approx 1-(1-\frac{logm}{N-1}) = \frac{logm}{N-1},$$

therefore

$$\bar{v}-v_i^* \approx (\frac{logm}{N-1})^{\frac{1}{\alpha}} c^{-\frac{1}{\alpha}} \approx A_1(\frac{logm}{N-1})^{\frac{1}{\alpha}}$$

and equality (L3-1) is established.

Equality L3-2.

First we choose the same α and δ as in the proof of Equality L3-1, then consider

$$\int_{\underline{v}}^{\bar{v}} F^N(p)dp = \int_{\underline{v}}^{\bar{v}-\delta} F^N(p)dp + \int_{\underline{v}-\delta}^{\bar{v}} F^N(p)dp. \qquad \text{(L3-3)}$$

We will show that

$$lim_{N\to\infty} N^{\frac{1}{\alpha}}\{\int_{\underline{v}-\delta}^{\bar{v}} F^N(p)dp\} = A_3$$

for some positive constant A_3.

Note that the previous δ is small enough such that the Taylor expansion is a good approximation, so

$$\int_{\underline{v}-\delta}^{\bar{v}} F^N(p)dp \approx \int_{\underline{v}-\delta}^{\bar{v}} [1-c(\bar{v}-p)^{\alpha}]^N dp.$$

Define $r = \frac{\bar{v}-p}{\delta}$, then

$$lim_{n\to\infty} N^{\frac{1}{\alpha}} \{ \int_{\underline{v}-\delta}^{\bar{v}} [1 - c(\bar{v}-p)^{\alpha}]^N dp \} \tag{L3-4}$$

$$= lim_{n\to\infty} N^{\frac{1}{\alpha}} \delta \{ \int_0^1 [1 - cr^{\alpha}]^N dr \}.$$

Now we partition [0,1] into l intervals $[r_0, r_1)$, $[r_1, r_2)$, $[r_2, r_3)$, ..., $[r_l, r_{l+1}]$, where $r_k := \frac{k}{N^{\frac{1}{\alpha}}}$ for $k = 0, 1, 2, ..., l+1$. Note that l is actually a function of the number of agents N. We approximate the integral $\int_0^1 (1 - cr^{\alpha})^N dr$ using a Riemann Sum by evaluating the function at the right end of each interval. Specifically, this gives

$$\int_0^1 (1 - cr^{\alpha})^N dr = lim_{N\to\infty} \sum_{k=1}^{l(N)} (1 - c\frac{k^{\alpha}}{N})^N * \frac{1}{N^{\frac{1}{\alpha}}}.$$

Plug this into equation (L3-4) and we get

$$lim_{N\to\infty} N^{\frac{1}{\alpha}} \{ \int_{\underline{v}-\delta}^{\bar{v}} [1 - c(\bar{v}-p)^{\alpha}]^N dp \} = lim_{N\to\infty} [\sum_{k=1}^{l(N)} (1 - c\frac{k^{\alpha}}{N})^N]$$

after some simple algebra, where δ is dropped since A_3 is arbitrary.

Since $(1 - c\frac{k^{\alpha}}{N})^N$ is weakly increasing in N, we have

$$\sum_{k=1}^{\infty} lim_{N\to\infty} [(1 - c\frac{k^{\alpha}}{N})^N] = \sum_{k=1}^{\infty} e^{-ck^{\alpha}},$$

which is equal to some positive constant by the comparison to $\sum_{k=1}^{\infty} e^{-k}$, so $\int_{\underline{v}-\delta}^{\bar{v}} F^N(p)dp$ has the same growth rate as $N^{-\frac{1}{\alpha}}$.

Now back to equality (L3-3), note that

$$lim_{N\to\infty} \frac{\int_{\underline{v}}^{\bar{v}-\delta} F^N(p)dp}{\int_{\underline{v}-\delta}^{\bar{v}} F^N(p)dp} \leq lim_{N\to\infty} \frac{c_1 F(\bar{v}-\delta)^N}{P\frac{1}{N^{\frac{1}{\alpha}}}} = 0,$$

where P is just some positive constant. Therefore $\int_{\underline{v}}^{\bar{v}} F^N(p)dp \approx A_2 \frac{1}{N^{\frac{1}{\alpha}}}$ as

N goes to infinity. □

Proof of Lemma 3 part (b).

First we prove that when the market has equal-share initial allocations, that is, $\overrightarrow{r} = (\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N})$, we have $lim_{N\to\infty}T_1 \leq 0$. This will imply that T_1 will be bounded above for any initial allocation $\overrightarrow{r}$ when N is sufficiently large, since T_1 is concave as we showed in the proof of Proposition 1.

Recall that

$$T_1 := \sum_{i=1}^{N}\{\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)\}.$$

Since agents' initial allocations are the same, we have

$$T_1 = N\{\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)\}. \tag{L3-5}$$

By the equality (T2-5) that we established in the proof of Theorem 2, we have

$$\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)$$

$$= \frac{\bar{v} - v_i^*}{N} - (\int_{v_i^*}^{\bar{v}} F^{N-1}(p)dp - \int_{\underline{v}}^{\bar{v}} F^{N}(p)dp) - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^{N}(p)dp.$$

Now note that

$$\int_{v_i^*}^{\bar{v}} F^{N-1}(p)dp - \int_{\underline{v}}^{\bar{v}} F^{N}(p)dp \geq \int_{\underline{v}}^{v_i^*} F^{N-1}(p)dp \geq 0,$$

and thus

$$\int_{v_i^*}^{\bar{v}}[1 - F(p)]pdG(p) - \int_{\underline{v}}^{v_i^*} F(p)pdG(p)$$

$$\leq \frac{\bar{v} - v_i^*}{N} - \frac{1}{N}\int_{\underline{v}}^{\bar{v}} F^{N}(p)dp$$

$$\approx \frac{1}{N}A_1(\frac{logN}{N-1})^{\frac{1}{\alpha}} - \frac{A_2N^{-\frac{1}{\alpha}}}{N}$$

by Lemma 3.

Therefore equation (L3-5) implies that

$$T_1 \leq A_1(\frac{logN}{N-1})^{\frac{1}{\alpha}} - A_2 N^{-\frac{1}{\alpha}},$$

where the right side goes to zero as N goes to positive infinity, so $lim_{N\to\infty} T_1 \leq 0$ is proved and thus T_1 is bounded above for any initial allocation vector $\overrightarrow{r}$.

Now we only need to prove that T_1 is bounded below for the market in the statement. Notice that Cramton et al. (1987) shows that efficiency can be achieved if $\beta = 0$ and N is large enough. By our discussion in the text after Proposition 1, this means $T_1 \geq 0$ when m is sufficiently large. □

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