A Theory of House Allocation and Exchange Mechanisms[*]

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Abstract

We study the allocation and exchange of indivisible objects without monetary transfers. In market design literature, some problems that fall in this category are the house allocation problem with and without existing tenants, and the kidney exchange problem. We introduce a new class of direct mechanisms that we call trading cycles with brokers and owners, and show that (i) each mechanism in the class is coalitionally strategy-proof and Pareto-efficient, and (ii) each coalitionally strategy-proof and Pareto-efficient direct mechanism is in the class. As corollaries, we obtain new characterizations in the aforementioned market design problems.

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1 Introduction

The theory and practical applications involving the allocation and exchange of indivisible resources without monetary transfers have recently been attracting attention of economists. Market designers have tailored new models and mechanisms to solve real-life problems such as the allocation of students to on-campus dormitory rooms at US colleges (cf. Abdulkadiroğlu and Sönmez, 1999) and exchanges of live donor kidney transplants (cf. Roth, Sönmez, and Ünver, 2004).

There are common features of these real-life problems. There is a group of agents each of whom would like to consume an indivisible object to which we will refer to as a house using the terminology coined by Shapley and Scarf (1974). Moreover, there is a group of houses to be distributed according to the agents' strict preferences over the houses. We will refer to such problems as house allocation and exchange problems. We study direct revelation mechanisms, that is, agents reveal their preferences over houses, and the mechanism assigns a house to each agent.

The direct mechanisms studied in the literature have two essential properties: Pareto-efficiency and coalitional strategy-proofness. Coalitional strategy-proofness means that no group of agents can jointly manipulate so that all of them weakly benefit from this manipulation, while at least one in the group strictly benefits. Such mechanisms are not only non-manipulable but also impose minimal computational costs on the participants and do not discriminate agents based on their ability to strategize and their access to information (cf. Vickrey, 1961, Dasgupta, Hammond, and Maskin, 1979, and Pathak and Sönmez, 2007).

We introduce a new class of direct mechanisms that we call trading cycles with brokers and owners, and show that (i) each mechanism in the class is coalitionally strategy-proof and Pareto-efficient, and (ii) each coalitionally strategy-proof and Pareto-efficient direct mechanism can be implemented through a mechanism from the class. Thus, we characterize the full class of relevant direct mechanisms, and lay down the structure of the house allocation and exchange problem. The trading-cycles-with-brokers-and-owners mechanisms can be used to solve practical design problems that were beyond the reach of the previously known mechanisms.

A trading-cycles-with-brokers-and-owners algorithm matches houses and agents in a sequence of rounds. At each round some agents and houses are matched and removed from the problem. At the beginning of the round, each previously unmatched house is controlled by an unmatched agent. We distinguish two forms of control over a house which we call ownership and brokerage (at any round, there is at most one broker and one brokered house). Each house points to the agent that controls it, and each agent points to his most preferred unmatched house. The only exception is the broker (if there is one) who points to his most preferred unmatched house other than the brokered house. In the resultant directed graph, there exists at least one exchange cycle. Each agent in each exchange cycle is matched with the house he points to.

The allocation of control rights in each round is fully determined by how agents and houses were matched prior to that round. The above-described procedure takes as given the mapping from partial matchings to control rights. Each such mapping that satisfies certain compatibility conditions determines a mechanism in our class.

The above class of mechanisms is built on the top-trading cycles idea attributed to David Gale by Shapley and Scarf (1974), and developed by Abdulkadiroğlu and Sönmez (1999), and Pápai (2000). The subclass of our mechanisms without brokers was introduced by Pápai (2000); it is the largest class of coalitionally strategy-proof and Pareto-efficient mechanisms previously known.

Against this background, our main innovation lies in introducing the brokerage control rights. Previously, to the best of our knowledge, only ownership control rights were studied in the context of house allocation and exchange. Recognizing the role of brokers in house allocation and exchange is crucial to obtaining the entire class of coalitionally strategy-proof and Pareto-efficient mechanisms. The introduction of brokers is also useful in some design problems.

As an example of a mechanism design problem in which brokerage rights are useful, consider a manager who assigns n tasks $t_1, ..., t_n$ to n employees $w_1, ..., w_n$ with strict preferences over the tasks. The manager wants the allocation to be Pareto-efficient with regard to the employees' preferences. Within this constraint, she would like to avoid assigning task t_1 to employee w_1. She wants to use a coalitionally strategy-proof direct mechanism, because she does not know employees' preferences. The only way to do it using the previously known mechanisms is to endow employees $w_2, ..., w_n$ with the tasks, let them find the Pareto-efficient allocation through a top-trading cycles procedure, such as Pápai's (2000) hierarchical exchange, and then allocate the remaining task to employee w_1. Ex ante each such procedure is unfair to the employee w_1. Using a trading-cycles-with-brokers-and-owners mechanism, the manager can achieve her objective without the extreme discrimination of the employee w_1. She makes w_1 the broker of t_1, allocates the remaining tasks among $w_2, ..., w_n$ (for instance she may make w_i the owner of t_i, $i = 2, ..., n$), and runs trading cycles with brokers and owners.

In our main result, we assume that all houses are social endowments, and hence there are no exogenous constraints on the allocation of control rights (cf. Hylland and Zeckhauser, 1979). For instance, at some universities, the dormitory rooms are treated as social endowments. At other universities however, some students, such as sophomores, have the right to stay in the room they lived in the preceding year. In kidney exchange, patients (interpreted as agents) come with a paired-donor (interpreted as a house) and have to be matched with at least their paired-donor. We derive corollaries of our main result for problems in which some houses are private endowments of agents and the participation in the mechanism has to be individually rational.

There are many studies that characterize desirable properties of house allocation and exchange through variants of top-trading cycles mechanisms. The most general class of mechanisms in the literature prior to our study was constructed by Pápai (2000). Her class characterizes coalitional strategy-proofness and Pareto-efficiency together with an additional property which she calls reallocation-proofness. A mechanism is reallocation-proof if there does not exist a profile of preferences, a pair of agents and a pair of preference misrepresentations such that (i) if both of them misrepresent their preferences, both of them weakly gain and one of them strictly gain by swapping their assignments, and (ii) if only one of them misrepresents his preferences, he cannot change his assignment. She also notes that the stronger property without condition (ii) conflicts with coalitional strategy-proofness and Pareto-efficiency. We do not use reallocation-proofness in our result.[1]

In matching and house allocation and exchange literature, the standard modeling approach has been to use strict preferences instead of the full preference domain. Participants are frequently allowed to submit only strict preference orderings to real-life direct mechanisms in various markets, such as dormitory room allocation, school choice, matching of interns and hospitals. As Ehlers (2002) shows "one cannot go much beyond strict preferences if one insists on efficiency and coalitional strategy-proofness." He characterizes coalitionally strategy-proof and Pareto-efficient mechanisms in the maximal subset of full preference domain such that such a mechanism exists. The full preference domain gives rise to an impossibility result, i.e., when agents can be indifferent among houses, there exists no mechanism that is coalitionally strategy-proof and Pareto-efficient. Under strict preferences, his class of mechanisms is a subclass of ours, and substantially different from the general class.[2]

The study of strategy-proof and Pareto-efficient mechanisms has a long tradition. Gibbard (1973) and Satterthwaite (1975) have shown under minor restrictions that all strategy-proof voting rules are dictatorial. Satterthwaite and Sonnenschein (1981) extended this result to public good economies with production, and Zhou (1991) extended it to pure public good economies. In social choice models, Dasgupta, Hammond and Maskin (1979) have proved that every Pareto-efficient and strategy-proof social choice rule is dictatorial. In exchange economies, Barberà and Jackson (1995) showed that strategy-proof mechanisms are Pareto-inefficient.

Even with additional structure, it has been difficult to characterize Pareto efficient and strategy-proof mechanisms that are non-dictatorial. Such characterizations have been obtained by Green and Laffont (1977) in decision problems with monetary transfers and quasi-linear utilities (cf. Vickrey, 1961, Clarke, 1971, Groves, 1973) and by Barberà, Gül, and Stacchetti (1993) in voting problems with single-peaked preferences (cf. Moulin, 1980, and Sprumont, 1991).[3]

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[1] Ma (1994), Svensson (1999), Ergin (2000), Miyagawa (2002), Ehlers, Klaus and Pápai (2002), Ehlers and Klaus (2004), Kesten (2004), Sönmez and Ünver (2006), and Ehlers and Klaus (2007) characterize subclasses of coalitionally strategy-proof, Pareto-efficient and reallocation-proof mechanisms.

[2] See Bogomolnaia, Deb and Ehlers (2005) for another characterization with indifferences.

[3] Sönmez (1999) studies generalized matching problems in which each agent is endowed with a good. The class of such problems non-trivially intersects with the class of house allocation and exchange problems studied in this paper. He shows that (i) there exists a Pareto-efficient, strategy-proof, and individually rational mechanism if, and only if, the core is nonempty and agents are indifferent between all core allocations. He also shows that any such mechanism is coalitionally strategy-proof (cf. Shapley and Scarf, 1974, Roth and Postlewaite, 1977, Roth, 1982, and Ma, 1994).

2 The Model

Let I be a set of **agents** and H be a set of **houses.** We use letters i, j, k to refer to agents and h, g, e to refer to houses. Each agent i has a **strict preference relation** over H, denoted by $\succ_i$.[4] Let $\mathcal{P}$ be the set of strict preference relations. Let $\succ = (\succ_i)_{i\in I} \in \mathcal{P}^{|I|}$ be a **preference profile**. For any $\succ \in \mathcal{P}^{|I|}$, and $J \subseteq I$, let $\succ_J = (\succ_i)_{i\in J}$ be the restriction of $\succ$ to J. A **house allocation problem** is denoted as a triple $\langle I, H, \succ\rangle$. We will assume that $|H| \geq |I|$ so that each agent is allocated a house. Each house allocation problem $\langle I, H, \succ\rangle$ that violates this assumption might be embedded in an augmented problem $\langle I, H', \succ'\rangle$ in which $H' \supseteq H$, the restriction of $\succ'$to houses in H coincides with $\succ$, and $|H'| \geq |I|$.

An outcome of a house allocation problem is a *matching.* To define a matching, let us start with a more general concept that we will use frequently. A **submatching** is an allocation of a subset of houses to a subset of agents, such that no two different agents get the same house. Formally, a submatching is a one-to-one and onto function $\sigma : J \to G$ such that $J \subseteq I$, $G \subseteq H$ and $|J| = |G|$. Let $\sigma(i)$ be the assignment of agent i under σ. Let $\mathcal{S}$ be the set of submatchings. For each $\sigma \in \mathcal{S}$ with $\sigma : J \to G$, let $I_\sigma \equiv J$ be the set of agents and $H_\sigma \equiv G$ be the set of houses matched under σ. Whenever it is convenient, we represent a submatching $\sigma \in \mathcal{S}$ as a set of matches,

$$\sigma = \{(i, \sigma(i))\}_{i\in I_\sigma}.$$

For any $h \in H$, let $\mathcal{S}_{-h} \subset \mathcal{S}$ be the set of submatchings $\sigma \in \mathcal{S}$ with $h \notin H_\sigma$, i.e. the set of submatchings at which h is unmatched. For each $h \in H_\sigma$ and $\sigma \in \mathcal{S}_{-h}$, $\sigma^{-1}(h) \in I_\sigma$ is the agent that got house h under σ, that is, $\sigma(i) = h$.

A **matching** is a submatching that matches all agents in I. Formally, a matching is a submatching $\mu \in \mathcal{S}$ such that $I_\mu = I$. Let $\mathcal{M} \subset \mathcal{S}$ be the set of matchings.

A **(direct) mechanism** is a systematic procedure that assigns a matching for each problem. Throughout the paper, we fix I and H, and thus, a problem is identified with its preference profile. Therefore, formally a mechanism is a function $\varphi : \mathcal{P}^{|I|} \longrightarrow \mathcal{M}$.

3 Coalitional Strategy-Proofness and Pareto Efficiency

In this section, we introduce essential properties of house allocation mechanisms.

A matching is **Pareto-efficient**, if there is no matching that makes everybody weakly better off, and at least one agent strictly better off. That is, a matching $\mu \in \mathcal{M}$ is Pareto-efficient if there exists no matching $\nu \in \mathcal{M}$ such that for all $i \in I$, $\nu(i) \succeq_i \mu(i)$, and for some $i \in I$, $\nu(i) \succ_i \mu(i)$. A mechanism is **Pareto-efficient**, if it finds a Pareto-efficient matching for every problem.

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[4] Let $\succeq_i$ be the induced weak preference relation, that is for any $g, h \in H$, $g \succeq_i h \iff g = h$ or $g \succ_i h$. A weak preference relation is a linear order on H, i.e. a binary relation on H that is antisymmetric, transitive, complete, and reflexive.

A mechanism is *coalitionally strategy-proof* if there is no group of agents that can misstate their preferences in a way such that each one in the group gets a weakly better house, and at least one agent in the group gets a strictly better house. Formally, a mechanism φ is **coalitionally strategy-proof** if for all $\succ \in \mathcal{P}^{|I|}$, there exists no $J \subseteq I$ and $\succ'_J \in \mathcal{P}^{|J|}$ such that

$$\begin{array}{lllr} \varphi\left[\succ'_J, \succ_{-J}\right](i) & \succeq_i & \varphi\left[\succ\right](i) & \forall i \in J \text{ and} \\ \varphi\left[\succ'_J, \succ_{-J}\right](j) & \succ_j & \varphi\left[\succ\right](j) & \exists j \in J. \end{array}$$

Coalitional strategy-proofness is dominant strategy incentive compatibility for a group and is a cooperative property. Therefore, it has a verifiability problem when agents communicate their private information and action plan to each other in a coalition in the standard non-cooperative settings. On the other hand, in our domain, it also has a non-cooperative interpretation. It is equivalent to the combination of two non-cooperative axioms, non-bossiness and strategy-proofness (Pápai, 2000). *Non-bossiness* (Satterthwaite and Sonnenschein, 1981) of a mechanism requires that when an agent misreports his preferences and gets the same assignment that he was getting under truthful revelation, then he cannot change the allocation regarding the other agents, either. *Strategy-proofness* of a mechanism means that the truthful revelation of preferences is a weakly dominant strategy.

Formally, a mechanism φ is **non-bossy** if for all $\succ \in \mathcal{P}^{|I|}$, $i \in I$, and $\succ'_i \in \mathcal{P}$,

$$\varphi\left[\succ'_i, \succ_{-i}\right](i) = \varphi\left[\succ\right](i) \quad \Rightarrow \quad \varphi\left[\succ'_i, \succ_{-i}\right] = \varphi\left[\succ\right].$$

Formally, a mechanism φ is **strategy-proof** if for all $\succ \in \mathcal{P}^{|I|}$, there exists no $i \in I$ and $\succ'_i \in \mathcal{P}$ such that

$$\varphi\left[\succ'_i, \succ_{-i}\right](i) \ \succ_i \ \varphi\left[\succ\right](i).$$

We state their relationship with the following lemma:

Lemma 1 *(Pápai, 2000) A house-allocation mechanism is coalitionally strategy-proof if and only if it is strategy-proof and non-bossy.*

There is another property that is closely related to coalitional strategy-proofness. A mechanism is *(Maskin-)monotonic* if whenever the preferences change in such a way that the set of houses better than the assigned house weakly shrinks for each agent, then the matching assigned by the mechanism does not change (Dasgupta, Hammond and Maskin, 1979). Formally, a mechanism φ is **Maskin-monotonic** if for all $\succ, \succ' \in \mathcal{P}^{|I|}$. and $i \in I$,

$$\{h \in H : h \succeq_i \varphi\left[\succ\right](i)\} \supseteq \{h \in H : h \succeq'_i \varphi\left[\succ\right](i)\} \quad \Rightarrow \quad \varphi\left[\succ'\right] = \varphi\left[\succ\right].$$

We also say that for any such $\succ'$ and $\succ$, profile $\succ'$ is a **monotonic extension of $\succ$ under φ**.

The following lemma states the relationship of Maskin monotonicity and coalitional strategy-proofness in our domain.[5]

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[5] The result obtains because our domain consists of strong preferences and is "rich" in the sense of Dasgupta, Hammond and Maskin (1979).

Lemma 2 *(Dasgupta, Hammond and Maskin, 1979).A house-allocation mechanism is Maskin-monotonic if and only if it is coalitionally strategy-proof.*

We will use these two equivalences in our proofs.

4 The Trading-Cycles-with-Brokers-and-Owners Algorithm

In this section, we introduce a new algorithm called *trading cycles with brokers and owners.* The assignment produced by our algorithm depends on the structure of control rights. Let us define this new concept first.

Definition. A **structure of control rights** (c, b) consists of

- A profile of **control functions** $c = \left(c_h : \mathcal{S}_{-h} \rightarrow I\right)_{h \in H}$ such that $c_h(\sigma) \in I - I_\sigma$.
- A **b-house function** $b : \mathcal{S} - \mathcal{M} \longrightarrow H \cup \{\emptyset\}$ such that for all $\sigma \in \mathcal{S} - \mathcal{M}$ if $|I_\sigma| = |I| - 1$, then $b(\sigma) = \emptyset$.[6]

Fix a proper submatching σ. If $\sigma \in \mathcal{S}_{-h}$ then agent $c_h(\sigma)$ is said to **control** (or **inherit**) house h at the σ. A house $h \in H - H_\sigma - \{b(\sigma)\}$ is called an occupied house, or owner's house, or **o-house** at σ (or shortly, σ-o-house), agent $c_h(\sigma)$ is called an **owner** of h at σ (or simply, σ-owner of h), and the pair $(c_h(\sigma), h)$ is called an **o-pair** at σ (or simply, σ-o-pair). If $b(\sigma) \neq \emptyset$ then the house $b(\sigma)$ is called the brokered house or **b-house** at σ (or, shortly, σ-b-house), the agent $c_{b(\sigma)}(\sigma)$ is called the **broker** at σ (or, shortly, σ-broker), and the pair $\left(c_{b(\sigma)}(\sigma), b(\sigma)\right)$ is called a **b-pair** at σ (or simply, σ-b-pair).

For any control rights structure (c, b), the assignment of houses to agents is determined by an iterative algorithm that we call the **trading-cycles-with-brokers-and-owners algorithm** (**TCBO algorithm** for short). In each round of the algorithm some agents are removed (matched).

In the description of the algorithm, we will use some graph theoretical concepts. The directed graphs constructed in the algorithm have unremoved agents and houses as the vertices; each house points to a single agent, and each agent points to a single house (thus determining the directed edges). A **cycle** of length n is a directed graph

$$h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow ...h^n \rightarrow i^n \rightarrow h^1$$

such that $i^\ell \in I$ points to $h^{\ell+1} \in H$ and h^ℓ points to i^ℓ for $\ell \in \{1, ..., n\}$, and $i^\ell \neq i^{\ell'}$, $h^\ell \neq h^{\ell'}$ for any $\ell, \ell' \in \{1, ..., n\}$, $\ell \neq \ell'$. Whenever we talk about cycles of length n, we will identify superscripts $n+1$ and 1. In each directed graph, there exists at least one cycle and no two cycles intersect.

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[6] In the sequel, we will develop compatibility conditions for the control rights structures.

> **Round $r = \mathbf{1}, \mathbf{2}, \ldots$ of the TCBO algorithm.** Let σ^{r-1} be the submatching of agents and houses removed before round r. Before the first round, the submatching of removed agents is empty, $\sigma^0 = \varnothing$.
>
> *Determination of intra-round trade graph.* Each unremoved house $h \in H - H_{\sigma^{r-1}}$ points to the agent who controls it at σ^{r-1}. If there exists a σ^{r-1}-*broker*, he points to his first choice σ^{r-1}-*o-house.* Every other unremoved agent $i \in I - I_{\sigma^{r-1}}$ points to his top choice house among the unremoved houses $h \in H - H_{\sigma^{r-1}}$.
>
> *Removal of Trading Cycles:* There exists at least one cycle. We remove each agent in each cycle by assigning him the house he is pointing to.
>
> *Stopping Rule.* We stop the algorithm if all agents are removed (matched). The resultant matching σ^r is then the outcome of the algorithm.

Since we assign at least one agent a house in every round, and since there are finitely many agents, the algorithm stops after finitely many rounds.

The terminology of owners and brokers is motivated by the trading analogy. In each round of the algorithm, an owner can either trade a house he controls for another house (in a cycle of several exchanges), or can leave in this round matched with a house he owns. A broker can trade the house he owns for another house (in a cycle of several exchanges), but cannot leave in this round matched with the house he brokers. One interpretation of this is that the owner can consume his house, but the broker cannot.

Our algorithm builds upon Gale's top-trading cycles idea (Shapley and Scarf, 1974) but differs in one important aspect from all other variants of the top-trading cycles algorithm such as top-trading cycles algorithm with newcomers (Abdulkadiroğlu and Sönmez, 1999), hierarchical exchange algorithm (Pápai, 2000), top-trading cycles algorithm for school choice (Abdulkadiroğlu and Sönmez, 2003) and top-trading cycles and chains algorithm (Roth, Sönmez, and Ünver, 2004). All these algorithms may be interpreted as TCBO in which all control rights are ownership rights and there are no brokers. In particular, in all these algorithms, in every round, all remaining agents point to their remaining top choice house. Thus, they refer to the forming exchange cycles as *top-trading cycles.* In TCBO algorithm, all remaining owners point to their remaining top choice house, except the broker. He points to his remaining top choice *o-house* and he does not point to the *b-house* even if it is his top remaining choice.

5 Examples

We start with an example of how the TCBO algorithm is executed.

Example 1. Execution of the TCBO algorithm. Let $I = \{i_1, i_2, i_3\}$ and $H = \{h_1, h_2, h_3\}$. Suppose the control rights structure is such that

- h_1 is owned by i_1 as long as i_1, h_1 are unmatched, is owned by i_2 when i_2, h_1 are unmatched and i_1 is matched, and is owned by i_3 when i_3, h_1 are unmatched and i_1, i_2 are matched,
- h_2 is owned by i_2 as long as i_2, h_2 are unmatched, is owned by i_1 when i_1, h_2 are unmatched and i_2 is matched, and is owned by i_3 when i_3, h_2 are unmatched and i_1, i_2 are matched,
- h_3 is controlled by i_3; he has the brokerage right as long as either i_1 and i_2 are unmatched and the ownership right when i_1 and i_2 are matched (notice that we do not need to specify who inherits h_3 when i_3 is matched, because i_3 may be matched only in a cycle that also contains h_3).

The above structure of control rights may be graphically represented as follows:

c_{h_1}	c_{h_2}	c_{h_3}
i_1	i_2	$i_3{}^{b}$
i_2	i_1	
i_3	i_3	

The ^{b} sign, above, next to i_3 in h_3's control right column, shows that h_3 is a b-house (when some agents other than i_3 who controls h_3 are unmatched). The preferences of the agents are given as follows:

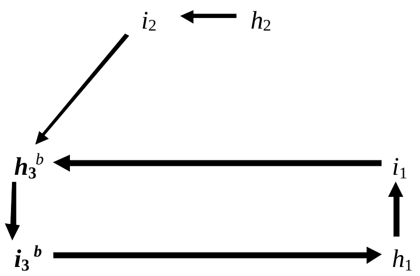
$$\text{agent 1: } h_3 \succ_{i_1} h_2 \succ_{i_1} h_1$$

$$\text{agent 2: } h_3 \succ_{i_2} h_2 \succ_{i_2} h_1$$

$$\text{agent 3: } h_3 \succ_{i_3} h_1 \succ_{i_3} h_2$$

We run the algorithm as follows:

Round 1. O-house h_1 points to $c_{h_1}(\varnothing) = i_1$, o-house h_2 points to $c_{h_2}(\varnothing) = i_2$, b-house $b(\varnothing) = h_3$ points to $c_{b(\varnothing)}(\varnothing) = i_3$. Agents i_1 and i_2 point to h_3 and broker i_3 points to his first choice o-house, that is h_1. This directed graph is given below:



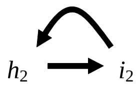
There exists one cycle

$$h_1 \to i_1 \to h_3 \to i_3 \to h_1,$$

and by removing it, we obtain

$$\sigma^1 = \{(i_1, h_3), (i_3, h_1)\}$$

Round 2. O-house h_2 points to $c_{h_2}(\sigma^1) = i_2$ and agent i_2 points to h_2. This directed graph is given below:



There exists one cycle $h_2 \to i_2 \to h_2$, and by removing it, we obtain

$$\sigma^2 = \{(i_1, h_3), (i_3, h_1), (i_2, h_2)\}.$$

This is a matching, since no agents are left.

We terminate the algorithm, the outcome of the mechanism is σ^2.

Remark. The mechanism (i.e., the mapping from preference profiles to assignments) of Example 1 is equivalent to the mechanism of Example 6 in Pápai (2000), though Pápai's algorithm differs considerably from our trading-cycles-with-brokers-and-owners algorithm. Pápai shows that the mechanism is coalitionally strategy-proof and Pareto-efficient but does not satisfy her reallocation-proofness condition (cf. the introduction for the definition of reallocation-proofness). Thus, this example shows that our class of mechanisms is more general than Pápai's class.

Example 2. TCBO with Persistent Brokers. The control rights structure (c, b) is said to have persistent brokers (or be strongly compatible) if, for any submatching $\sigma \in \mathcal{S} - \mathcal{M}$, we have:

C1. [**Persistence of ownership**] If i owns h at σ, and i and h are unmatched at $\sigma' \supset \sigma$, then i owns h at σ'.

C2. [**No ownership for brokers**] If k is a broker at σ, then k does not own any house at σ.

SC3. [**Persistence of brokerage**] If agent k brokers house e at σ, and k, e and an agent $i \neq k$ are unmatched at $\sigma' \supset \sigma$, then k brokers e at σ'.

In Section 7, we will show that each TCBO with persistent brokers is coalitionally strategy-proof and Pareto-efficient. Let us now look at what these strong compatibility conditions imply through the following problem. Suppose $I = \{i_1, i_2, ..., i_5\}$ and $H = \{h, h_2, ..., h_5\}$. Let (c, b) be a strongly compatible control rights structure. The structure regarding house h is represented through the following graph:

This graph can be interpreted as follows: House h is initially inherited by i_1, that is $c_h(\varnothing) = i_1$. If agent i_1 is assigned house h_2 or h_3 then the control right of h is inherited by agent i_2, when he is assigned h_4, the control right of h is inherited by i_3, and finally, when he is assigned h_5, the control right of h is inherited by agent i_4. That is,

$$\begin{aligned} c_h(\{(i_1, h_2)\}) &= c_h(\{(i_1, h_3)\}) = i_2, \\ c_h(\{(i_1, h_4)\}) &= i_3, \\ c_h(\{(i_1, h_5)\}) &= i_4. \end{aligned}$$

This tree structure shows the persistence of o-pairs and t-pairs (C1 and SC3). The ^{b} sign next to agent i_3 shows that house h becomes a b-house when it is inherited by i_3, moreover the broker status of i_3 persists until h_3 is removed. For strong compatibility (C2), we need i_3 be the last inheritor in subtree of c_h following submatching $\{(i_1, h_4)\}$ for all $h \in \{h_2, h_3, h_5\}$.

Similarly as we follow other submatchings, we obtain

$$\begin{aligned} c_h(\{(i_1, h_2), (i_2, h_3)\}) &= c_h(\{(i_1, h_3), (i_2, h_2)\}) = i_4. \\ c_h(\{(i_1, h_2), (i_2, h_4)\}) &= c_h(\{(i_1, h_3), (i_2, h_4)\}) \\ &= c_h(\{(i_1, h_2), (i_2, h_5)\}) = c_h(\{(i_1, h_3), (i_2, h_5)\}) = i_3. \end{aligned}$$

In either case, h becomes a b-house, but its control rights are given to different agents, i_4 or i_3 respectively. We can interpret the remainder of the control rights structure of house h, similarly.

6 The Trading-Cycles-with-Brokers-and-Owners Mechanism

We are ready to formally define our mechanism class. We will define compatibility of control right structures by maintaining Conditions C1-C2 of Example 2, and relaxing condition SC3.

Definition (Compatibility of Control Right Structures). A control rights structure (c, b) is **compatible**, if for any submatching $\sigma \in \mathcal{S} - \mathcal{M}$,

C1. [**Persistence of ownership**] If i owns h at σ, and i and h are unmatched at $\sigma' \supset \sigma$, then i owns h at σ'.

C2. [**No ownership for brokers**] If k is a broker at σ, then k does not own any house at σ.

C3. [**Limited persistence of brokerage**] If k brokers e at σ, and agent $j \neq k$ and house $g \neq e$ are unmatched at σ, then either

- there are no agents who own a house both at σ and $\sigma \cup \{(j, g)\}$, or
- there is exactly one agent i who owns a house both at σ and $\sigma \cup \{(j, g)\}$, and (i) agent i owns e at $\sigma \cup \{(j, g)\}$, and (ii) at every $\sigma' \supset \sigma \cup \{(j, g), (i, e)\}$ at which k is unmatched, k owns all houses that i owns at σ, or
- there is at least one agent who own a house both at σ and $\sigma \cup \{(j, g)\}$, and k brokers e at $\sigma \cup \{(j, g)\}$.

Each *compatible* pair (c, b) induces a **trading-cycles-with-brokers-and-owners mechanism** (**TCBO mechanism** for short). We denote it as $\psi^{c,b}$. Its outcome is found through the TCBO algorithm that was introduced earlier. All control rights structures discussed in Examples 1-3 are compatible, hence the induced mechanisms are TCBO.

Under the compatibility conditions, a b-pair may not persist as larger submatchings are fixed. This is the only difference from the strong compatibility conditions introduced in Example 2 for persistent brokers.

If no previous owners are left as one more agent is matched, then the b-pair may not persist.

Also, a broker k can enter a *broker-to-heir transition* only when exactly one previous owner i remains unmatched at the larger submatching with one more agent matched. After the broker k loses his brokerage privilege, the ex-b-house is owned by the owner i. However, we need a protection for the ex-broker k in this case, since he could have used his brokerage privilege in a trade earlier to get a house that i owned. So, if i gets the ex-b-house and leaves, then the inheritance system defined in C3 guarantees that the ex-broker k owns the owner i's houses that he owned at the instant the broker k went into transition. Thus, we refer to the ex-broker as an *heir to i* in this case.

On the other hand, if there are more than 1 previous broker unmatched at the larger submatching, then the b-pair persists, and this is the *limited persistence of brokerage* defined in C3.The following example illustrates the limited persistence of b-pairs.

Example 3. Limited persistence of b-pairs. Consider an environment with four agents: $I = \{j, i_1, i_2, k\}$ and four houses: $H = \{h_1, h_1', h_2, e\}$, and a TCBO mechanism $\psi^{c,b}$ whose control rights structure (c, b) is illustrated by the following table explained below (cf. also Example 1 for the use of tables):

c_{h_1}	$c_{h_1'}$	c_{h_2}	c_e
i_1	i_1	i_2	$k^{\ b}$
j	j	i_1	$(i_1, h_1): \quad i_2^{\text{ broker}\rightarrow\text{heir}}$
i_2	i_2	j	j
k	k	k	k

In particular,

- Houses h_1, h_1' are owned by agent i_1, when he is matched the unmatched of the two houses is owned by j, then i_2, and k. House h_2 is owned by i_2, then i_1, j, and k.
- Agent k has the brokerage right over e initially (i.e., at the empty submatching). He remains the broker as long as he is unmatched with one exception. The broker-to-heir transition occurs after the assignment (i_1, h_1). At this point i_2 is the only remaining owner left from the previous round (hence C3 is satisfied). Then, agent i_2 becomes the owner of e. The superscript, broker→heir, next to i_2 denotes this transition in the control rights structure of house e.

Notice, that C2 is satisfied as broker k is the last inheritor of all o-houses.

Remark. The above compatible control rights structure (c, b) cannot be interpreted as a strongly compatible structure with persistent brokers (introduced in Example 2):

Proof. By way of contradiction, let us assume that there is a TCBO mechanism with persistent brokers ϕ that produces the same allocation as $\psi^{c,b}$ for each profile of agents' preferences.

Let us define the following classes of preference profiles. For any submatching $\sigma \in \mathcal{S} - \mathcal{M}$ and house $h \in H - H_\sigma$, let $\mathbf{P}[\sigma, h] \subset \mathcal{P}^{|I|}$ be the set of preference profiles such that for any $\succ \in \mathbf{P}[\sigma, h]$,

- for all $i \in I_\sigma$, $\sigma(i) \succ_i g$ for all $g \in H - \{\sigma(i)\}$, that is, each i matched at σ ranks house $\sigma(i)$ as his first choice, and
- for all $i \in I - I_\sigma$, $h \succ_i g \succ_i g'$ for all $g \in H - H_\sigma - \{h\}$, and $g' \in H_\sigma$, that is, each i unmatched at σ ranks (i) house h as his first choice, and (ii) the houses matched at σ lower than the houses unmatched at σ.

For any submatching $\sigma \in \mathcal{S}$ and houses $h, h' \in H - H_\sigma$, let $\mathbf{P}[\sigma, h, h'] \subset \mathcal{P}^{|I|}$ be the set of preference profiles such that for any $\succ \in \mathbf{P}[\sigma, h, h']$,

- for all $i \in I_\sigma$, $\sigma(i) \succ_i g$ for all $g \in H - \{\sigma(i)\}$, that is, each i matched at σ ranks house $\sigma(i)$ as his first choice, and
- for all $i \in I - I_\sigma$, $h \succ_i h' \succ_i g \succ_i g'$ for all $g \in H - H_\sigma - \{h, h'\}$, and $g' \in H_\sigma$, that is, each i unmatched at σ ranks (i) house h as his first choice, (ii) house h' as his second choice, and (iii) the houses matched at σ lower than the houses unmatched at σ.

First, notice that at the empty submatching, k is the *broker* of e in the TCBO mechanism with persistent brokers ϕ. It is so, because e is not owned by any agent at the empty submatching $\varnothing$ as $(\phi[\succ])^{-1}(e) = \left(\psi^{c,b}[\succ]\right)^{-1}(e)$ varies with $\succ \in \mathbf{P}[\varnothing, e]$. Hence, there is an agent who has the brokerage right over e and it must be k as $\phi[\succ](k) = \psi^{c,b}[\succ](k) = g$ for any $\succ \in \mathbf{P}[\varnothing, e, g]$, $g \in \{h_1, h_1', h_2\}$.

Second, consider the submatching $\sigma = \{(i_1, h_1)\}$ and a preference profile $\succ \in \mathbf{P}[\sigma, e, h_1']$. In mechanism ϕ, agent k would continue to be the broker of e at σ and thus

$$\phi[\succ](k) = h_1'.$$

However,

$$\psi^{c,b}[\succ](k) = h_2$$

and thus $\phi \neq \psi^{c,b}$ for any TCBO mechanism with persistent brokers ϕ. QED

Remark: When $|I| = 3$, brokers are persistent in compatible control rights structures.

Proof. First notice that with only two agents left unmatched, having one owner and one broker is equivalent to having one owner and one agent with no rights. That is, even if the broker controls the b-house, the owner will definitely get it if he wants it. Thus, this is observationally equivalent to the owner also owning the "b-house".

Next, suppose there is a broker and the other two agents are owners. Even if the broker loses his status after one of the owners get matched, by the reasoning in the previous paragraph, it will be equivalent to broker remaining as a broker. Thus, in this case, *strong compatibility conditions* are equivalent to *compatibility conditions.* Since with three agents, this is the only case when a broker can lose his status under the compatible control rights structures, the brokers are persistent. QED

The above remark and Theorems 1-4 proved next, imply that

Corollary 1 *If $|I| = 3$, then a mechanism is coalitionally strategy-proof and Pareto-efficient if and only if it is TCBO with persistent brokers.*

In the next section, we will show that a mechanism is coalitionally strategy-proof and Pareto-efficient if and only if it is equivalent to a TCBO mechanism.

7 The Main Result

In this section, we first show that trading-cycles-with-brokers-and-owners mechanism satisfy coalitional strategy-proofness and Pareto-efficiency (Theorems 1-3). Theorem 4 shows that any coalitionally strategy-proof and Pareto-efficient direct mechanism is TCBO. The proof of this result is constructive, given any coalitionally strategy-proof and Pareto-efficient mechanism ψ it shows how to construct the corresponding control rights structure (c, b), and verifies the resultant TCBO $\psi^{c,b} = \psi$.

Theorem 1 *The trading-cycles-with-brokers-and-owners algorithm produces a Pareto-efficient matching for any control rights structure that satisfies C2.*

Proof of Theorem 1: Consider the execution of the TCBO algorithm. Observe that each agent removed in Round 1 gets his top choice house, possibly except a broker removed. If there exists a broker, the b-house is his top choice, and he is removed in Round 1, then the broker receives his second choice. Observe that by C2, a broker can only be removed together with a b-house, since he does not own any house. Thus, the b-house is assigned to an agent who had the b-house as his top choice, hence if the broker were to be assigned the b-house, he would make this other agent worse off. Thus, nobody removed in Round 1 can be made better off without making somebody removed in Round 1 worse off. Observe that each agent removed in Round 2 gets his remaining top choice, possibly except a broker. If there exists a broker, the b-house is his remaining top choice, and he is removed in Round 2, then he receives his second remaining choice. In this case, the b-house is also assigned to an agent in Round 2 (by C2), and this agent prefers it more than any house assigned in the second round. Thus, nobody removed in Round 2 can be made better off without making somebody removed in Round 1 or 2 worse off. We continue iteratively, showing that the outcome of the TCBO algorithm is Pareto-efficient. **QED**

Next, we make three observations about the TCBO algorithm:

Observation 1. At any round r of the algorithm, the control rights structure (i.e., current set of o-pairs, control rights owners, and the possible b-pair), depends only on the removed submatching σ^r. In other words, if two preference profiles $\succ$ and $\succ'$ induce the same σ^r at round r, then they induce the same control rights structure.

Pápai (2000) uses the name "twin inheritance rule" to refer to an analogous observation about her subclass of the class of mechanisms studied in this paper.

Observation 2. If an agent i is unmatched at a round r of the algorithm under preference profiles $[\succ_i, \succ_{-i}]$ and $[\succ'_i, \succ_{-i}]$, then the same submatching is left before r, that is $\sigma^{r-1}[\succ_i, \succ_{-i}] = \sigma^{r-1}[\succ'_i, \succ_{-i}]$.

Observation 3. If an agent i is unmatched at a round r of the algorithm under preference profiles $[\succ_i, \succ_{-i}]$ and $[\succ_i', \succ_{-i}]$, then the control rights structure is the same under $[\succ_i, \succ_{-i}]$ and $[\succ_i', \succ_{-i}]$.

In particular, an agent cannot affect, by submitting different preferences, when he becomes a broker, enters broker-to-heir transition, and becomes an owner.

Next, we will prove that TCBO mechanisms (that is, the mappings that find the outcome of each problem through a compatible control rights structure using the TCBO algorithm) are also strategy-proof and non-bossy, implying through Lemma 1 that they are coalitionally strategy-proof.

Theorem 2 *Every trading-cycles-with-brokers-and-owners mechanism is* strategy-proof.

Proof of Theorem 2: Let $\psi^{c,b}$ be a TCBO mechanism. Let $\succ$ be a preference profile. We fix an agent $i \in I$. We will show that i cannot benefit by submitting $\succ_i' \neq \succ_i$ while the other agents submit $\succ_{-i}$. Let s be the round i leaves (with house h) at $\succ_i$ and s' be the time i leaves (with h') at $\succ_i'$ in the algorithm. We will consider two cases.

Case $s \leq s'$. At round s, same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by Observation 3. If i is not a broker at time s under $\succ_i$, then, by submitting $\succ_i$, agent i gets the best object among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

Assume now that i is a broker at time s under $\succ_i$. Let e be the b-house at time s. If e is not agent i's top choice house remaining under $\succ_i$, then by submitting $\succ_i$, agent i gets the best object among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

It remains to consider the situation in which e is broker i's top choice remaining house, and to show that i cannot get e by submitting the profile $\succ_i'$. For an argument through contradiction, assume that under $\succ_i'$ agent i leaves at round s' with house e.

Notice that under $\succ_i$, there is an agent j who is matched with house e at time s, because i is a broker and he leaves in the same cycle in round s. At this time, j is an owner of some o-house h_j, and e is his top choice house. By Observation 3, the control rights structure at round s is the same under both $\succ_i$ and $\succ_i'$. Hence, i is also a broker at time s after submitting $\succ_i'$, and j is an owner of h_j. Moreover, j's top choice is still house e. That means that under $\succ_i'$ agent j will stay unmatched till $s'+1$. Since agent i leaves with e at s', he cannot be the broker of e at this round, because a broker cannot leave with the b-house, while another owner j is unmatched. Thus, there is a round $s'' \in \{s+1, ..., s'\}$ at which agent i stops being the broker of e. Since e is still unmatched at this round, there is a broker-to-heir transition between $s''-1$ and s'' (by C3). Because j is an owner of h_j at both $s''-1$ and s'', he would have inherited e at s'' (by C3). Then, however, j would have left with e at s'', as e is j's top choice among houses left at s (and hence those left at s''). A contradiction.

Case $s > s'$. At round s', same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by Observation 3.

Consider round s' at both $\succ_i$ and $\succ'_i$. Under $\succ'_i$, agent i points to house $h' = h^1$ that points to agent i^1 that points to ... that points to object h^n that points to agent $i = i^n$ (and this cycle leaves at round s'). If the cycle is trivial ($n = 1$) and h' points back to i, then (i, h') is an o-pair. Since o-pairs persist by C1, it will be an o-pair at $s > s'$, and thus at round s, agent i would leave with a house at least as good as h'.

In the sequel, assume that there is at least one other agent i^n in the cycle (that is $n \geq 2$).

If all pairs $\left(i^\ell, h^\ell\right)$ are o-pairs, $\ell = 1, ..., n$, then the chain $h' = h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow ... \rightarrow h^n \rightarrow i$ will stay in the system as long as i is in the system (by persistency of o-pairs implied through C1). Thus, at round s agent i would leave with a house at least as good as h' under $\succ_i$.

If $\left(i^\ell, h^\ell\right)$ is a b-pair for some $\ell = 1, ..., n$, then the chain $h' = h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow ... \rightarrow h^n \rightarrow i$ will stay in the system as long as $\left(i^\ell, h^\ell\right)$ continues to be a b-pair (since there are no other b-pairs and the o-pairs persist by C1). If it is a b-pair at round s under $\succ_i$, then we are done, since the same cycle would have formed. Thus suppose that at a round $s'' \in \{s' + 1, ..., s\}$ broker i^ℓ loses his broker status. Because $n \geq 2$, agent $i^{\ell+1}$ is an owner both rounds $s'' - 1$ and s''. Hence, the loss of brokerage status means that i^ℓ enters broker-to-heir transition. We must then have $n = 2$ (since by C3, only 1 previous owner can remain unmatched during broker-to-heir transition). There are two cases: either i^1 owns $h^1 = h'$ and h^2 (and $i^2 = i^\ell$ is the heir) or $i^2 = i$ owns h^1 and h^2. In the former case, i^1 who wants h^2, will leave with it at round s'' under $\succ_i$, and i will inherit $h^1 = h'$ at $s'' + 1$ by the definition of an inheritance system and by C3. In the latter case, i owns $h^1 = h'$ already at round s''. In both cases, at $s \geq s''$ agent i can only leave with a house at least as good as h' under $\succ_i$. **QED**

Theorem 3 *Every trading-cycles-with-brokers-and-owners mechanism is* non-bossy*.*

The proof of this theorem is in Appendix A. In lieu of a heuristic argument, we state and prove the following weaker proposition in which the assumption C3 is replaced by a stronger assumption SC3 (formulated in Example 2).

Proposition 1 *Every TCBO mechanism with control rights structures satisfying C1, C2 and SC3 is* non-bossy*.*

Proof of Proposition 1. Let $\psi^{c,b}$ be a TCBO mechanism with persistent brokers (that is, (c, b) is strongly compatible and satisfies C1, C2 and SC3). Let $\succ$ be a preference profile. We fix an agent $i_* \in I$. We will show that i_* cannot change the allocation of the other agents by submitting $\succ'_{i_*} \neq \succ_{i_*}$ and obtaining the same house

$$\psi^{c,b}\left[\succ'_{i_*}, \succ_{-i_*}\right](i_*) = \psi^{c,b}[\succ](i_*) = h_*.$$

Let $\succ' = \left[\succ'_{i_*}, \succ_{-i_*}\right]$. Let s be the round i_* leaves (with house h_*) under $\succ$ and s' be the round i leaves (with h_*) under $\succ'$ in the TCBO algorithm. Without loss of generality, suppose $s' \geq s$. By Observation 3, we have $\sigma^r(\succ') = \sigma^r(\succ)$ for all $r \in \{1, 2, ..., s-1\}$.

We will show that any cycle removed in a round $r \geq s$ of the algorithm under $\succ$ will be removed in some round r' of the algorithm under $\succ'$. The argument will be by an induction with respect to r.

For $r = s$, the same houses and agents are in the market under both $\succ$ and $\succ'$. By Observation 3, any cycle that is formed in round s under $\succ$, will form under $\succ'$ in round s, as well, except possibly the cycle in which i_* leaves in round s under $\succ$. Let us denote this cycle by $C_* = h_* \to i^1_* \to h^2_* \to ...h^n_* \to i_* \to h_*$. We show that cycle C_* forms under $\succ'$ in some round r'. By C1 and SC3, the same chain $h_* \to i^1_* \to h^1_* \to ...i^{n-1}_* \to h^n_* \to i_*$ will remain in the algorithm, as long as i_* is unmatched. Moreover, since i_* gets h_* under $\succ'$, he will point to h_* in round $s' > s$ and the same cycle C_* will form and will be cleared.

For the inductive step, fix round $r > s$ and assume that all cycles cleared in all rounds before r under $\succ$ are also cleared in some round of the algorithm under $\succ'$. Let

$$C = h^1 \to i^1 \to h^2 \to ...h^n \to i^n \to h^1$$

be a cycle that formed in round r under $\succ$. Let r' be the earliest round that one of the agents in C was removed under $\succ'$, and let i^ℓ be an agent removed at r' under $\succ'$. By the inductive step, i^ℓ gets house $h^{\ell+1}$ or worse under $\succ'$, and thus $h^{\ell+1}$ is removed at $r'' \leq r'$.[7] Let $\Sigma = \sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$. House $h^{\ell+1}$ is controlled by $i^{\ell+1}$ at Σ by C1 and SC3. Let $j^{\ell+1}$ be the agent that controls house $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$. By C1 and SC3, we either have $i^{\ell+1} = j^{\ell+1}$ or $j^{\ell+1}$ is matched at $\sigma^{r-1}(\succ)$. The latter case cannot happen. Indeed, $j^{\ell+1}$ is matched at $\sigma^{r''}(\succ)$ in a cycle containing $h^{\ell+1}$. If $j^{\ell+1}$ was matched at $\sigma^{r-1}(\succ)$ then the inductive assumption implies that also $h^{\ell+1}$ is matched at $\sigma^{r-1}(\succ)$, a contradiction. Hence, $i^{\ell+1} = j^{\ell+1}$ owns $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$, and $i^{\ell+1}$ is removed at r'' under $\succ'$. Thus, $r'' = r'$. By repeating the argument we conclude that $i^{\ell+2}$ owns $h^{\ell+2}$ at $\sigma^{r'-1}(\succ')$ and is removed at r' under $\succ'$, etc. Thus, all agents i^ℓ, $\ell = 1, ..., n$, are removed at r', while controling h^ℓ and desiring $h^{\ell+1}$. Thus, the cycle C is removed at r' under $\succ'$. **QED**

Finally, we state the converse of Theorems 1 – 2, completing this section:

Theorem 4 ***(Implementation Result)*** *If a mechanism is coalitionally strategy-proof and Pareto-efficient then it is a trading-cycles-with-brokers-and-owners mechanism.*

The proof of this theorem is in Appendix B. In the proof we fix a coalitionally strategy-proof and Pareto-efficient direct mechanism φ and construct a TCBO mechanism $\psi^{c,b}$ that implements φ.

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[7] The superscripts are modulo n, that is $n + 1 = 1$.

We first construct the candidate control rights structure (c, b) using the set of profiles $\mathbf{P}^*[\sigma, h^1] = \cup_{h \in (H - H_\sigma) - \{h^1\}} \mathbf{P}[\sigma, h^1, h]$ where $\mathbf{P}[\sigma, h^1, h]$ was introduced in Example 3. It is straightforward to see that in a TCBO mechanism ψ, a σ-o-house h owned by i satisfies the property

$$\psi[\succ](i) = h \text{ for all } \succ \in \mathbf{P}^*[\sigma, h].$$

We use this property to define candidate o-houses and owners. A house $h \in H$ is a candidate o-house at $\sigma \in \mathcal{S}$ if it is unmatched at σ and the agent $\varphi[\succ]^{-1}(h)$ is constant across all $\succ \in \mathbf{P}^*[\sigma, h]$; then the agent $\varphi[\succ]^{-1}(h)$ is the candidate owner of h at σ. Furthermore, a σ-b-house e satisfies the property

$$\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e) \text{ for some } \succ, \succ' \in \mathbf{P}^*[\sigma, e],$$

and broker k satisfies

$$\varphi[\succ](k) = h \text{ for all } \succ \in \mathbf{P}[\sigma, e, h].$$

We use this properties to define candidate b-houses and brokers. A house $e \in H$ is a candidate b-house at $\sigma \in \mathcal{S}$ if it is unmatched at σ and there exist some $\succ, \succ' \in \mathbf{P}^*[\sigma, e]$, such that $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$. An agent k is a candidate broker of house e at σ, if e is a candidate b-house at σ and for all $\succ \in \mathbf{P}^*[\sigma, h]$, house $\varphi[\succ](k)$ is the second choice of k in $\succ_k$.

We then show that the resultant control rights structure (c, b) is well defined and compatible. This is the hardest part of the proof. Finally, we show that the induced TCBO mechanism $\psi^{c,b}$ implements φ, that is, $\varphi[\succ] = \psi^{c,b}[\succ]$ for each preference profile $\succ$.

8 Market Design, Private Endowments, and Individual Rationality

In this section, we will introduce some market design problems which have relevance in both theory and application. These are modifications of the house allocation problem. In each such problem, there are also agents with private endowments. We will give characterizations in these domains using our main result.

8.1 House Allocation Problem with Existing Tenants

Let $I^E \subseteq I$ be the set of agents each of whom **initially occupies** a house. We refer to set I^E as the set of **existing tenants**. Let the house h_i be owned by each agent $i \in I^E$. Let $H^O = \{h_i\}_{i \in I^E} \subseteq H$ be the set of **initially occupied houses**. Preferences etc. are defined as before. Let $\succ$ be a preference profile. **A house allocation problem with existing tenants** is a list $\langle H, I, (i, h_i)_{i \in I^E}, \succ \rangle$ (Abdulkadiroğlu and Sönmez, 1999). Matchings and mechanisms are defined in the same manner as before. We fix H, I and $(i, h_i)_{i \in I^E}$ so that a problem is defined just through its preference profile $\succ$.

This problem has an ownership structure in its fundamentals unlike the house allocation problem. It is modeled after the real-life dormitory allocation problems in the US college campuses. In each such college, at the beginning of the academic year, there are new senior, junior, sophomore students, each of whom already occupies a room from the last academic year. There are vacated rooms by the graduating class and there are new freshmen who would like to obtain a room, though they do not currently occupy any.

Besides Pareto efficiency and coalitional strategy-proofness, another important property of mechanisms of this problem domain is individual rationality. A matching is **individually rational**, if it assigns the existing tenants a house that is at least as good as their own house. Formally a matching μ is individually rational if

$$\mu(i) \succeq_i h_i \qquad \forall i \in I^E.$$

A mechanism is **individually rational** if it always selects an individually rational matching.

If a mechanism is not individually rational, an existing tenant may opt out from trading and the mechanism may result with an inefficient outcome. We will introduce an individually rational version of the TCBO mechanism. Let $\psi^{c,b}$ be a TCBO mechanism. Clearly, if we do not have any restrictions on the (c, b) pair other than compatibility, an existing tenant can receive a house worse than his initial occupied house. We make the following **individual rationality restrictions** on (c, b):

- For each $i \in I^E$, let $c_{h_i}(\varnothing) = i$, that is, every existing tenant inherits his initially occupied house at the beginning of the algorithm.
- For each $i \in I^E$, $h_i \neq b(\varnothing)$, that is, an initially occupied house cannot be a b-house at the beginning of the algorithm.

A TCBO mechanism is referred to as an **individually rational TCBO mechanism** (**IR-TCBO mechanism**) if its control rights function – b-house function pair satisfies the individual rationality restrictions in addition to compatibility conditions.

A few remarks will be useful for the b-house function in this case. First of all, since an o-pair persists (by C1), no initially occupied house can be a b-house. Moreover, since an owner cannot be a broker (by C2), and each existing tenant owns his initially occupied house, an existing tenant cannot be a broker in the TCBO algorithm.

First, we state the following equivalence result:

Proposition 2 *In* ***house allocation problems with existing tenants****, a TCBO mechanism is* individually rational *if and only if it is equivalent to an IR-TCBO mechanism.*

Proof of Proposition 2: Let ψ be an individually rational TCBO mechanism. Let $i \in I^E$ and $\succ \in \mathbf{P}[\varnothing, h_i]$. Thus, $\psi[\succ](i) = h_i$ by individual rationality of ψ. Two cases are possible:

If h_i is a $\varnothing$-o-house under ψ, then clearly agent i $\varnothing$-owns h_i.

If h_i is a $\varnothing$-b-house under ψ, then agent i should be the $\varnothing$-owner of all other houses so that he can get his initially occupied house h_i at any such profile $\succ \in \mathbf{P}[\varnothing, h_i]$. But then $I^E = \{i\}$, because i would have gotten h_j at all preference profiles in $\mathbf{P}[\varnothing, h_j]$ for all of the other existing tenants $j \neq i$, contradicting ψ is individually rational . With $I^E = \{i\}$, this TCBO mechanism is equivalent to the one at which agent i $\varnothing$-owns all houses. Thus, ψ is equivalent to an IR-TCBO mechanism.

Conversely, suppose ψ is an IR-TCBO mechanism. Then, each initially occupied house is $\varnothing$-owned by its existing tenant. Thus for each $i \in I^E$, $\psi[\succ](i) \succeq_i h_i$ for any preference profile $\succ$, showing that ψ is individually rational. **QED**

We will use IR-TCBO mechanisms in our next result:

Proposition 3 *In* ***house allocation problems with existing tenants****, a mechanism is* individually rational, Pareto-efficient *and* coalitionally strategy-proof *if and only if it is equivalent to an IR-TCBO mechanism.*

Proof of Proposition 3: Let φ be an individually rational, Pareto-efficient and coalitionally strategy-proof mechanism for the allocation with existing tenants problem. By Theorem 4 there exists a compatible (c, b) pair such that $\varphi = \psi^{c,b}$. By Proposition 2, $\psi^{c,b}$ should be equivalent to an IR-TCBO mechanism.

Converse of it follows from Theorems 1-3. **QED**

8.2 Kidney Exchange with Good Samaritan Donors

Another important real-life problem is the *kidney exchange problem* (Roth, Sönmez and Ünver, 2004). We add *good Samaritan donors* to this model following Sönmez and Ünver (2006): Let Π be a set of **transplant patients**, each of whom needs a kidney transplant. Let Δ be the set of **live donors**, each of whom is willing to donate a kidney. Each transplant patient $p \in \Pi$ has **at least one paired-donor** $d_{p,1} \in \Delta$ or he can have more $\left\{d_{p,1}, ..., d_{p,n(p)}\right\} \subseteq \Delta$. All paired-donors of a patient are willing to donate a kidney if and only if the patient himself gets a kidney transplant. Moreover, at most one of the patient's paired-donors will be matched.

All donors besides the paired-donors, are **good samaritan (GS) donors** (or, *altruistic donors*), they are willing to donate a kidney no matter who receives it. Thus, we treat the donors as objects and the patients as the agents. Patients have strict preferences over Δ. Let $\succ$ be a preference profile. **A kidney exchange problem with GS-donors** is given as a list $\left(\left\{p, d_{p,1}, ..., d_{p,n(p)}\right\}_{p\in\Pi}, \Delta, \succ\right)$. We fix $\left\{p, d_{p,1}, ..., d_{p,n(p)}\right\}_{p\in\Pi}$ and Δ. Thus, we denote a kidney exchange problem with GS-donors through its preference profile $\succ$.

In the US, GS-donors have been the driving force behind kidney exchange since 2006. Beginning with the 5-way kidney exchange with a GS-donor conducted in Johns Hopkins Transplant Center, many regional programs such as Alliance for Paired Donation (centered in Toledo, Ohio) and New England Program for Kidney Exchange (centered in Newton, Massachusetts) have used GS-donors in majority of kidney exchanges that they conducted.

A note will be useful about coalitional strategy-proofness in this context. The doctors of the patients are the ones who have the information about patients' preferences over kidneys and it is well-known that doctors (or transplant centers) themselves manipulate the system, if it will benefit their patients.[8] Hence, if kidney exchange mechanisms are coalitionally strategy-proof, a doctor (or a transplant center) will not be able to manipulate the mechanism on behalf of his (or its) patients without hurting at least one of them.

Observe that this problem is an instance of the house allocation problem with existing tenants. The only difference is that there are no *newcomers* (i.e., patients without any donors).

When there are no newcomers, there are no brokers in the IR-TCBO mechanisms. By definition, brokers cannot own any donors (houses), however each patient in our model owns at least one donor. Therefore, we refer to IR-TCBO mechanisms as **individually rational hierarchical exchange mechanisms** (in this domain), since they reduce to Pápai's (2000) mechanism with the individual rationality constraint. Here is our result of this subsection:

Proposition 4 *In* ***kidney exchange problems with good Samaritan Donors****, a mechanism is* individually rational*,* Pareto-efficient *and* coalitionally strategy-proof *if and only if it is an individually rational hierarchical exchange mechanism.*

Proof of Proposition 4: Let φ be an individually rational, Pareto-efficient and coalitionally strategy-proof mechanism for the kidney exchange problem with GS-donors. This problem is an instance of the house allocation problem with existing tenants with no newcomers. By Proposition 3, there exists an individually rational and compatible (c, b) pair such that $\varphi = \psi^{c,b}$. Since each patient owns* at least one donor under φ, by Lemma 11 (see Appendix B), there exists no broker* for φ.[9] By equivalence of φ to $\psi^{c,b}$, brokers and brokers* are equivalent at each $\sigma \in \mathcal{S} - \mathcal{M}$. Thus, there is no broker for $\psi^{c,b}$ implying that $\psi^{c,b}$ is an IR-hierarchical exchange mechanism.

Converse of it follows from Theorems 1-3 and Proposition 3. **QED**

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[8] For example, deceased-donor queue procedures are frequently gamed, by physicians acting as advocates for their patients. In particular, on July 29, 2003 two Chicago hospitals settled a Federal lawsuit alleging that some patients had been fraudulently certified as sicker than they were to move them up on the liver transplant queue (Warmbir, 2003).

[9] See Appendix B for the formal definition of owner* and broker*.

A Appendix: Proof of Theorem 3

Proof of Theorem 3: Let $\psi^{c,b}$ be a TCBO mechanism. Fix an agent $i^* \in I$ and two preference profiles $\succ = [\succ_{i_*}, \succ_{-i_*}]$ and $\succ' = \left[\succ'_{i_*}, \succ_{-i_*}\right]$ such that

$$h_* = \psi^{c,b}[\succ'](i_*) = \psi^{c,b}[\succ](i_*).$$

Let s be the round i_* leaves (with house h_*) submitting $\succ_{i_*}$ and s' be the time i_* leaves (with h_*) submitting $\succ'_{i_*}$. By symmetry, it is enough to consider the case $s \leq s'$. In order to show that

$$\psi^{c,b}[\succ](j) = \psi^{c,b}[\succ'](j) \text{ for all } j,$$

we will prove the following stronger statement:

Hypothesis: If a cycle of agents $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms and is removed at round r when preferences $\succ$ were submitted, then either

- same cycle $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms when preferences $\succ'$ are submitted, or
- $n = 2$ and two cycles $h^2 \to i^1 \to h^2$ and $h^1 \to i^2 \to h^1$ form when preferences $\succ'$ are submitted, or
- $n = 1$ and there exists agent $j \neq i^1$ and house $h \neq h^1$ such that the cycle $h \to i^1 \to h^1 \to j \to h$ forms when preferences $\succ'$ are submitted

Whenever in the proof we encounter cycles of length n, the superscripts on houses and agents will be understood to be modulo n, that is $i^{n+1} = i^1$ and $h^{n+1} = h^1$.

By Observation 3, the above hypothesis is true for any $r < s$. The proof for $r \geq s$ will proceed by induction over the round r.

Initial step. Consider $r = s$. Under $\succ$, house h_*^1 points to agent $i_* = i_*^1$ points to house $h_* = h_*^2$ that points to agent i_*^2 that points to ... that agent i_*^n that points to house h_*^1, and the cycle

$$h_*^1 \to i_*^1 \to h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1$$

is removed at round s. Observation 3 implies that the same houses and agents are in the market at time s under both $\succ$ and $\succ'$ and that all agents from $I_{\sigma^s(\succ)} - \{i_*^1, ..., i_*^n\}$ are matched by $\sigma^s(\succ')$ in the same way as in $\sigma^s(\succ)$.

Observation 3 also implies that the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ forms at round s under preferences $\succ'$.

If all pairs $\left(i_*^\ell, h_*^\ell\right)$, for all $\ell \in \{2, ..., n\}$, are $\sigma^s(\succ)$-o-pairs, then they are $\sigma^s(\succ')$-o-pairs and the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ will stay in the system as long as i_*^1 is in the system (by persistency of o-pairs through C1). Thus, at s' all agents $i_*^1, ..., i_*^n$ would leave with same houses as under $\succ$.

If $n > 1$, and $\left(i_*^\ell, h_*^\ell\right)$ is a b-pair for some $\ell \in \{2, ..., n\}$, then the chain $h_*^2 \rightarrow ... \rightarrow h_*^n \rightarrow i_*^n \rightarrow h_*^1 \rightarrow i_*^1$ will stay in the system as long as $\left(i_*^\ell, h_*^\ell\right)$ continues to be a b-pair. If $\left(i_*^\ell, h_*^\ell\right)$ continues to be a b-pair till round s' under $\succeq$, then the initial step is proved. Otherwise, there is a round $s'' \in \{s+1, ..., s'\}$ such that agent i_*^ℓ has brokerage right over h_*^ℓ at rounds $s, ..., s''-1$ but not at round s''. By C3, $n = 2$ and $i_*^{\ell+1}$ owns h_*^ℓ at $\sigma^{s''}(\succ')$ because he owns $h_*^{\ell+1}$at both $\sigma^{s''-1}(\succ')$ and $\sigma^{s''}(\succ')$. As $i_*^{\ell+1}$ top preference is then h_*^ℓ, he will leave with it at s''. By C3, agent i_*^ℓ will inherit $h_*^{\ell+1}$ at $s''+1$ and will be matched with it. This case ends the proof of the the inductive hypothesis for $r = s$.

Inductive step. Now, take any round $r > s$ such that $\sigma^r(\succ) - \sigma^{r-1}(\succ)$ is non-empty, and assume that the inductive hypothesis is true for all rounds up to $r-1$. Consider agents and houses

$$h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow ... \rightarrow h^n \rightarrow i^n \rightarrow h^1$$

who form a cycle at round r under $\succ$. Since all agents but i^* have same preferences in both profiles $\succ$ and $\succ'$, so do agents $i^1, ..., i^n$.

Let r' be the earliest round in which one of these agents is matched under $\succ'$, and let i^ℓ be an agent matched at r'. The argument will be divided into ten claims, the first of which follows directly from the inductive hypothesis.

Claim 1. Suppose that a house h forms a cycle with at least two agents under $\succ$. Then:

- If agent $i \in I_{\sigma^{r-1}(\succ)}$ belongs to the cycle of house h under $\succ'$, then i belongs to the cycle of h under $\succ$.
- If house h' belongs to the cycle of h under $\succ'$, then h' belongs to the cycle of h under $\succ$.

Claim 2. Suppose $\Sigma = \sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, and i owns h at Σ, and h belongs to a cycle at r'' under $\succ'$ and at r under $\succ$. If j controls h at $\sigma^{r''-1}(\succ')$ and is unmatched at Σ, then i is in one cycle with h at $\sigma^{r''-1}(\succ')$.

Proof of Claim 2: If $i = j$, then the result is true. Assume $i \neq j$. Then j does not control h at Σ. Thus, j brokers it at $\sigma^{r''-1}(\succ')$ and hence the $\sigma^{r''-1}(\succ')$-cycle of h contains j' and house h' that j' owns. By Claim 1, $j' \notin I_{\sigma^{r-1}(\succ)}$ and $h' \notin H_{\sigma^{r-1}(\succ)}$, and thus $j' \notin I_\Sigma$ and $h' \notin H_\Sigma$. By C3, j' owns h at Σ, and C1 implies that $i = j'$. QED

Claim 3. Suppose that $n > 1$ and agents $j, i^1, ..., i^n$ and house h^1 are unmatched at $\Sigma = \sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, j and $h^{\ell+1}$ are part of a cycle matched at round $r'' \leq r'$ under $\succ'$. If agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$, then, under $\succ'$, $i^{\ell+1}$ is matched at round r' and $h^{\ell+1}$ is matched at round r'.

Proof of Claim 3: If j owns $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$ then C1 implies that both j and $i^{\ell+1}$ own $h^{\ell+1}$ at Σ. Hence, $i^{\ell+1}=j$, and he is matched at r'' under $\succ'$. Because r' is the earliest round one of the agents $i^1,...,i^n$ is matched, it must be that $r'=r''$ and the claim is true.

If j brokers $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$, then let σ_0 be a minimal submatching in

$$\left\{\sigma\in S:\sigma^{r''-1}(\succ')\subseteq\sigma\subseteq\Sigma\right\}$$

at which j is not a broker of $h^{\ell+1}$. Let $j'\neq j$ belong to the cycle of $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$. Then j' is an owner of a house h' at $\sigma^{r''-1}(\succ')$. Because $n>1$, Claim 1 gives $j'\notin I_{\sigma^{r-1}(\succ)}$ and $h'\notin H_{\sigma^{r-1}(\succ)}$. Thus, (j',h') is an o-pair at $\sigma_0\subseteq\Sigma$. By C3, agent j' becomes the owner of $h^{\ell+1}$ at σ_0, and thus by persistency $\left(j',h^{\ell+1}\right)$ is an o-pair at Σ. Thus, $j'=i^{\ell+1}$ and he is matched at r'' in the same cycle as $h^{\ell+1}$. A fortiori, $r''=r'$. QED

Claim 4. Assume i^ℓ is a $\sigma^{r-1}(\succ)$-owner. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round then they are matched with each other.

Proof of Claim 4: House $h^{\ell+1}$ is agent i^ℓ's top choice among houses unmatched at $\sigma^{r-1}(\succ)$. By the inductive assumption, there exists r^* such that $\sigma^{r-1}(\succ)\subseteq\sigma^{r^*}(\succ')$, and thus $\psi^{c,b}[\succ']\left(i^\ell\right)$ is weakly worse than $h^{\ell+1}$. Hence, at the round i^ℓ is matched he points to $h^{\ell+1}$ (and then is matched with it) or a worse house (and then $h^{\ell+1}$ was matched earlier). QED

Claim 5. Suppose that agents $j,i^1,...,i^n$ and house $h^{\ell+1}$ are unmatched at $\Sigma=\sigma^{r-1}(\succ)\cup\sigma^{r''-1}(\succ')$, $i^{\ell+1}$ brokers $h^{\ell+1}$ at $\sigma^{r-1}(\succ)$, agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$, and j and $h^{\ell+1}$ are part of a cycle matched at round $r''\leq r'$ under $\succ'$. Then, under $\succ'$, $h^{\ell+1}$ is matched at round $r'=r''$.
Claim 6. Furthermore, either the inductive hypothesis is true, or $i^{\ell+1}$ is matched at round r'.

Proofs of Claim 5 and Claim 6: For convenience, assume $\ell=n$ that is $\ell+1=1$. Since, i^1 is a broker while matched under $\succ$, hence $n>1$, and we can use Claim 1. Moreover, if $i^1=j$, then i^1 controls h^1 at $\sigma^{r''-1}(\succ')$, and thus he is matched at r''. Because r' is the earliest round one of the agents $i^1,...,i^n$ is matched, it must be that $r'=r''$ and the claim is true. Hence, assume $i^1\neq j$.

Notice that

- If j controls h^1 at Σ then i^1 does not.
- If j does not control h^1 at Σ then j brokers it at $\sigma^{r''-1}(\succ')$ and hence the cycle of h^1 contains a $\sigma^{r''-1}(\succ')$-owner j' and a house h' owned by j'. By Claim 1, $j'\notin I_{\sigma^{r-1}(\succ)}$ and $h'\notin H_{\sigma^{r-1}(\succ)}$, and thus j' and h' are not matched at Σ. By C3, j' owns h^1 at Σ, and hence i^1 cannot broker it then.

In either case, i^1 does not broker h^1 at Σ. Thus, C3 implies that

(*) $n = 2$, and i^2 owns h^1 at Σ, and i^1 will inherit h^2 if i^2 is matched with h^1 at Σ.

If $i^2 \neq j$, then Claim 2 implies that i^2 is in the $\sigma^{r'-1}(\succ')$-cycle of h^1, and thus $r' = r''$.

If $i^2 = j$, Claim 4 implies that i^2 is matched at r'' in the cycle

$$h^1 \to i^2 \to h^1.$$

Thus, $r'' = r'$ and h^1 is matched at round r' under $\succ'$. That proves Claim 5.

In order to prove Claim 6, let r^1 be the time i^1 is matched, and r^2 be the time h^2 is matched under $\succ'$. By Claim 7 (whose proof depends on Claim 5 but not on Claim 6), $r^2 \leq r^1$. Let j^2 be the agent controlling h^2 at $\sigma^{r^2-1}(\succ')$. By Claim 1, $j^2 \notin I_{\sigma^{r-1}(\succ)}$. Let

$$\Sigma' = \sigma^{r-1}(\succ) \cup \sigma^{r^2-1}(\succ').$$

If $r^2 > r'$ then i^2 is matched to h^1 at $\Sigma' \supseteq \sigma^{r^2-1}(\succ') \supseteq \sigma^{r'}(\succ')$. By (*), i^1 owns h^2 at Σ'. By Claim 2, i^1 is in one cycle with h^2 at r^2.

If $r^2 \leq r'$ then i^2, the $\sigma^{r-1}(\succ)$-owner of h^2, is unmatched at $\sigma^{r^2-1}(\succ')$ and hence owns h^2 at Σ'. By Claim 2, i^2 is in one cycle with h^2 at $\sigma^{r^2-1}(\succ')$. That means that $r^2 = r'$. Let i be the agent controlling h^1 at $\sigma^{r'-1}(\succ')$. Since the cycle of h^1 contains i^1 and i^2, and i^2 gets h^1, hence $i \neq i^2$. Because, i^2 owns h^1 and h^2 at Σ', C3 implies that i inherits h^2 if i^2 is matched with h^1 at Σ'. By (*), i^1 inherits h^2 if i^2 is matched with h^1 at Σ, and hence also at Σ'. Thus, $i^2 = i$ belongs to the cycle of h^1 under $\succ'$. QED

Claim 7. Assume i^ℓ is a $\sigma^{r-1}(\succ)$-broker. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round, then they are matched with each other.

Proof of Claim 7: Notice that $n > 1$ and for notational convenience assume that $\ell = 1$. By the inductive assumption, There exists r^* such that $\sigma^{r-1}(\succ) \subseteq \sigma^{r^*}(\succ')$. Thus, if (i^1, h^2) does not satisfy the claim then the top preference of i^1 must be h^1, the house he brokers at $\sigma^{r-1}(\succ)$, and i^1 must get h^1 under $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r'-1}(\succ')$. Notice that j is matched in the same cycle as h^1 at round r' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}(\succ)}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}(\succ)}$. Thus, agents $j, i^1, ..., i^n$ and house h^1 are unmatched at the submatching $\Sigma = \sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$. By Claim 5, h^1 is matched at r', and by Claim 4, i^n gets h^1, a contradiction. QED

Claim 8. If $n = 1$ then either $h^1 \to i^1 \to h^1$ form a cycle under $\succ'$, or there exists an agent j and a house h such that $h \to i^1 \to h^1 \to j \to h$ form a cycle under $\succ'$.

Proof of Claim 8: Claim 2 implies that h^1 is matched at round $r'' \leq r'$ when preferences are $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r''-1}(\succ')$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Two cases are possible about j:

Case $j \in I_{\sigma^{r-1}(\succ)}$. Then $j \neq i^1$ and the inductive assumption and $h^1 \notin H_{\sigma^{r-1}(\succ)}$ imply that

- j is matched at $\sigma^{r-1}(\succ)$ in a cycle $h \to j \to h$ (for some house $h \neq h^1$), and
- there exists agent i such that the cycle $h \to i \to h^1 \to j \to h$ is matched at $\sigma^{r''}(\succ')$.

To finish the proof of the current case, it remains to be shown that $i = i^1$.

- If $i \in I_{\sigma^{r-1}(\succ)}$ then the inductive assumption implies that i is matched with h^1 under $\succ'$, and hence $i = i^1$.
- If $i \notin I_{\sigma^{r-1}(\succ)}$ then i and i^1 are unmatched at $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, and by persistency of o-pairs, i^1 owns h^1 at this submatching. Notice that j is an owner of h at $\sigma^{r-2}(\succ)$, and hence at $\sigma^{r-2}(\succ) \cup \sigma^{r''-1}(\succ')$. Thus, i must have been a broker of h at $\sigma^{r''-1}(\succ')$ and stopped being a broker at a submatching σ between $\sigma^{r''-1}(\succ')$ and $\sigma^{r-2}(\succ) \cup \sigma^{r''-1}(\succ')$. Because there might be only one broker at any submatching, j is an owner of h^1 at $\sigma^{r''-1}(\succ')$. Thus j is an owner at σ, and inherits h when i loses the broker status. When j is matched with h, the broker transition rules (C3) imply that i becomes the owner of h^1. Hence, i is the owner of h^1 at $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$ as is i^1. Thus, $i = i^1$.

Case $j \notin I_{\sigma^{r-1}(\succ)}$. Then, agents j, i^1 are unmatched at the submatching $\Sigma = \sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, and by persistency of o-pairs, i^1 owns h^1 at this submatching. Hence, either

- $j = i^1$ controls h^1 at $\sigma^{r''-1}(\succ')$, or
- $j \neq i^1$ is a broker of h^1 at $\sigma^{r''-1}(\succ')$, and loses the brokerage right at some submatching σ between $\sigma^{r''-1}(\succ')$ and $\sigma^{r-1}(\succ) \cup \sigma^{r''}(\succ')$.

In the former subcase, i^1 is matched at r'' as h^1 is matched then. Thus, $r'' = r'$, and a fortiori i^1 is matched with h^1 and hence owns h^1 at r'. The claim is then proved.

In the latter subcase, let $j' \neq j$ be an agent matched in the same cycle as h^1 at $\sigma^{r''-1}(\succ')$. Then j' is an owner of a house h' at $\sigma^{r''-1}(\succ')$.

We have $j' \notin I_{\sigma^{r-1}(\succ)}$ and $h' \notin H_{\sigma^{r-1}(\succ)}$, as otherwise the inductive assumption and $h^1 \notin H_{\sigma^{r-1}(\succ)}$ would imply that j' is matched at $\sigma^{r-1}(\succ)$ in a cycle $h' \to j' \to h'$, and there would exist an agent i such that the cycle $h' \to i \to h^1 \to j' \to h'$ is matched at r'' (under $\succ'$). The inductive assumption would further imply that h^1 is matched to i at $\succ$ and thus $i^1 = i$. Since j is in the cycle of h^1 and $j \neq j'$ and $j \neq i^1$, we would obtain a contradiction showing that $j' \notin I_{\sigma^{r-1}(\succ)}$.

Thus j' and h' are unmatched at $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, and hence they are unmatched at σ. Thus, C3 implies that agent j' is the owner of h^1 at σ, and by persistency through C1, (j', h^1) is an o-pair at $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$. Thus, $i^1 = j'$ and he is matched with h^1 at $r'' = r'$. Thus, at r' the cycle in which i^1 is matched is $h' \to i^1 \to h^1 \to j \to h'$. QED

Claim 9. Suppose $n > 1$. If $i^{\ell+1}$ is $\sigma^{r-1}(\succ)$-owner, then i^{ℓ} is matched with $h^{\ell+1}$ at $\sigma^{r'}(\succ')$, and $i^{\ell+1}$ is matched at $\sigma^{r'}(\succ')$.

Proof of Claim 9: By Claim 2, house $h^{\ell+1}$ is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house $h^{\ell+1}$ at $\sigma^{r''-1}(\succ')$. Notice that j is matched in the same cycle as $h^{\ell+1}$ at round r'' under $\succ'$. Since $h^{\ell+1} \notin H_{\sigma^{r-1}(\succ)}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}(\succ)}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$, and Claim 3 yields that $i^{\ell+1}$ is matched at $r'' = r'$ and then Claim 2 shows that i^{ℓ} is matched with $h^{\ell+1}$. QED

Claim 10. If $i^{\ell+1}$ is a $\sigma^{r-1}(\succ)$-broker, then either the inductive hypothesis is true or i^{ℓ} is matched with $h^{\ell+1}$ at $\sigma^{r'}(\succ')$, and $i^{\ell+1}$ is matched at $\sigma^{r'}(\succ')$.

Proof of Claim 10: For convenience let us assume that $\ell = n$ and $\ell + 1 = 1$. Agent i^n is a $\sigma^{r-1}(\succ)$-owner because $n > 1$ and i^1 is a $\sigma^{r-1}(\succ)$-broker. By Claim 2, house h^1 is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house at $\sigma^{r''-1}(\succ')$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}(\succ)}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}(\succ)}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}(\succ) \cup \sigma^{r''-1}(\succ')$. Claim 5 yields $r'' = r'$ and thus Claim 4 shows that i^n is matched with h^1 under $\succ'$. Claim 6 ends the proof. QED

Claim 8 proves the inductive hypothesis for cycles of length $n = 1$ and Claim 9 and Claim 10 applied recursively prove the hypothesis for cycles of length $n > 1$. This ends the proof of the theorem. **QED**

B Appendix: Proof of Theorem 4 (Implementation Result)

Let φ be a coalitionally strategy-proof and Pareto-efficient mechanism (fixed throughout Appendix B). We are to prove that φ may be represented as TCBO. We will first construct the candidate control rights structure (c, b) and then show that the induced TCBO mechanism $\psi^{c,b}$ is equivalent to φ.

In the construction of the control rights structure we will use the following notation (partially introduced already in Example 3). Let $\sigma \in \mathcal{S} - \mathcal{M}$, $n \geq 0$ and $h^1, h^2, ..., h^n \in H - H_\sigma$, and $i \in I$.

$\mathbf{P}_i[\sigma, h^1, ..., h^n]$ is the set of preferences $\succ_i$ of agent i such that

- if $i \in I_\sigma$ then

$$\sigma(i) \succ_i g$$

for all $g \in H - \{\sigma(i)\}$,

- if $i \in I - I_\sigma$ then
  $$h^1 \succeq_i h_i^2 \succeq ... \succeq_i h^n \succ_i g \succ_i g'$$
  for all $g \in H - H_\sigma - \{h^1, ..., h^n\}$, and $g' \in H_\sigma$.

$\mathbf{P}[\sigma, h^1, ..., h^n] \subset \mathcal{P}^{|I|}$ is the Cartesian product of $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ over all $i \in I$.
$\mathbf{P}^*[\sigma, h^1] = \cup_{h \in (H - H_\sigma) - \{h^1\}} \mathbf{P}[\sigma, h^1, h]$.

When σ is fixed, we will occasionally write $\langle h^1, ..., h^n, ...\rangle$ instead of $\mathbf{P}_i[\sigma, h^1, ..., h^n]$.

We say that a house $h \in H$ is an **o-house* at** $\sigma \in \mathcal{S}$ (or simply, σ-o-house*) if it is unmatched at σ and the agent $\varphi[\succ]^{-1}(h)$ is constant across all $\succ \in \mathbf{P}^*[\sigma, h]$. We refer to the agent $\varphi[\succ]^{-1}(h)$ as the **owner* of** h **at** σ (or, simply σ-owner*). We refer to the pair $\left(\varphi[\succ]^{-1}(h), h\right)$ as an **o-pair* at** σ (or, simply σ-o-pair*). We say that (i, h) is a **strong o-pair* at** σ (or simply, strong σ-o-pair*), if for all $\succ \in \mathbf{P}[\sigma, h]$, we have $\varphi[\succ](i) = h$. Observe that strong ownership* implies ownership*.

We say that a house $e \in H$ is a **b-house* at** $\sigma \in \mathcal{S}$ (or simply, σ-b-house*) if it is unmatched at σ and there exist some $\succ, \succ' \in \mathbf{P}^*[\sigma, e]$, such that $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$. We say that agent k is the **broker* of** e **at** σ (or simply, σ-broker*), if e is a σ-b-house* and for all $\succ \in \mathbf{P}^*[\sigma, h]$, house $\varphi[\succ](k)$ is the second choice of k in $\succ_k$. We refer to the pair (k, e) as a **b-pair* at** σ (or simply, σ-b-pair*).

We will show that the starred terms determine a compatible control right structure (c, b), and hence determine a TCBO mechanism $\psi^{c,b}$. The proof will be finished when we show that $\varphi = \psi^{c,b}$.

Notice that if φ is TCBO and i is an owner at σ then i is an owner* at σ, similarly for broker*. Thus, owners* and brokers* are candidate owners and brokers in the TCBO we will construct.

Following Papai (2000), we say that j envies i at $\succ$ if

$$\varphi[\succ](i) \succ_j \varphi[\succ](j).$$

Two lemmas proven in Pápai (2000) will be useful.

Lemma 3 *(Pápai, 2000). For any agents $i, j \in I$, and preferences $\succ \in \mathcal{P}^{|I|}$, $\succ_j^* \in \mathcal{P}$, if j envies i at $\succ$ and $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$, then*

$$\varphi[\succ](i) \succ_i \varphi\left[\succ_j^*, \succ_{-j}\right](i).$$

Lemma 4 *(Pápai, 2000). For any agents $i, j \in I$, and preferences $\succ \in \mathcal{P}^{|I|}$, $\succ_j^* \in \mathcal{P}$ if j envies i at $\succ$ and $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$, then there exists $\succ_i^* \in \mathcal{P}$ such that*

$$\varphi\left[\succ_i^*, \succ_j^*, \succ_{-\{i,j\}}\right](i) = \varphi[\succ](j).$$

The following is an immediate corollary of strategy-proofness and Lemma 4:

Corollary 2 *For any agents $i, j \in I$, and preferences $\succ \in \mathcal{P}^{|I|}$, $\succ_j^* \in \mathcal{P}$ if j envies i at $\succ$ and $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$, then*

$$\varphi\left[\succ_j^*, \succ_{-j}\right](i) \succeq_i \varphi[\succ](j).$$

To show that the starred terms introduced above determine a compatible control rights structure, we will prove a sequence of lemmas. We start with lemmas determining relations between ownership* and brokerage* at a fixed submatching $\sigma \in \mathcal{S} - \mathcal{M}$.

B.1 "Intratemporal" Lemmas (Fixed Submatching)

Lemma 5 *Let $\sigma \in \mathcal{S} - \mathcal{M}$. For any $i \in I_\sigma$, $h \in H - H_\sigma$*

$$\varphi[\succ](i) = \sigma(i) \text{ for all } \succ \in \mathbf{P}[\sigma, h].$$

Proof of Lemma 5. Suppose that an agent in $i \in I_\sigma$ does not get $\sigma(i)$ at $\varphi[\succ]$. Then we can create a matching by assigning all agents in $I - I_\sigma$ that get a house in H_σ a house in $H - H_\sigma$ that was assigned to an agent in I_σ, all other agents j in $I - I_\sigma$ the house $\varphi[\succ](j)$, and all agents j in I_σ the house $\sigma(j)$. Since each agent in $I - I_\sigma$ ranks houses in H_σ lower than houses in $H - H_\sigma$ and each agent in I_σ ranks his σ-house as his first choice, this matching Pareto-dominates $\varphi[\succ]$, contradicting φ is Pareto-efficient. **QED**

Lemma 6 *Let $\sigma \in \mathcal{S} - \mathcal{M}$. If $e, h \in H - H_\sigma$, then there exists $i \in I - I_\sigma$ such that $\varphi[\succ](i) = e$ for all $\succ \in \mathbf{P}[\sigma, e, h]$.*

Proof of Lemma 6: Suppose not. Let $\succ, \succ' \in \mathbf{P}[\sigma, e, h]$ be such that $\varphi[\succ](i) = e$ and $\varphi[\succ'](i') = e$ for some $i' \neq i$. Without loss of generality, we can assume that $\succ$ and $\succ'$ differ only in preferences of a single agent $j \in I - I_\sigma$. Let $g = \varphi[\succ](j)$. We can assume that

$$\succ_i \in \langle e, h, g, ...\rangle$$

because (i) if other agents submit $\succ_{-i}$ then Maskin monotonicity implies that the allocation does not depend on how i ranks houses below e, and (ii) if other agents submit $\succ'_{-i}$, then i cannot obtain e by moving g up in his ranking; thus there is an agent other than i who obtains e.

Let $g' = \varphi[\succ'](j)$. By non-bossiness, $g \neq g'$. By strategy-proofness, $j \neq i, i'$ and hence $e \neq g, g'$. By Maskin monotonicity we can further assume that

$$\succ_j \in \langle e, h, g, g', ...\rangle \text{ and } \succ'_j \in \langle e, h, g', g, ...\rangle,$$

and that the only difference between $\succ_j$ and $\succ'_j$ is in relative ranking of g and g'.

By Pareto efficiency, an agent from $I - I_\sigma$ is allocated h at $\succ$. Let $k = \varphi[\succ]^{-1}(h)$ and fix

$$\succ^*_k \in \langle e, g, h, ... \rangle .$$

We prove the lemma in several claims:

Claim 1. $\varphi[\succ'](i) = g$ and $\varphi[\succ'](k) = h$.

Proof of Claim 1. Since agent j envies i at $\succ$, Corollary 2 and strategy-proofness of φ imply that i gets at least g at $\succ'_i$. Since i gets e at $\succ$ and something worse at $\succ'$, Lemma 3 implies that agent j cannot envy i at $\succ'$. Thus $\varphi[\succ'](i) \neq h$. Since $i \neq i'$ hence $\varphi[\succ'](i) \neq e$ and thus $\varphi[\succ'](i) = g$.

Since agent j envies k at $\succ$, Corollary 2 and strategy-proofness of φ imply that k gets at least g at $\succ'_k$. Since i' gets e at $\succ'$ and something worse at $\succ$, Lemma 3 implies that agent j cannot envy i' at $\succ$. Thus $\varphi[\succ](i') \neq h$ and $i' \neq k$. Thus, k does not get e at $\succ'$. Because $k \neq i$ does not get g at $\succ'$, hence $\varphi[\succ'](k) = h$. QED

Claim 2. If $\succ^*_i \in \langle h, e, g, ... \rangle$ then $\varphi[\succ^*_i, \succ_{-i}](i) = h$ and $\varphi[\succ^*_i, \succ_{-i}](j) = g$.

Proof of Claim 2. Consider $\succ^*_i \in \langle h, e, g, ... \rangle$. By strategy-proofness, $\varphi[\succ^*_i, \succ_{-i}](i) \succsim^*_i e$. Everybody else ranks e over h. Thus by Pareto efficiency of φ, i should get h at $[\succ^*_i, \succ_{-i}]$.

On the other hand, Claim 1 implies that i gets g at $\succ'$, and by Maskin monotonicity of φ, $\varphi\left[\succ^*_i, \succ'_{-i}\right] = \varphi[\succ']$. Thus, j gets g' at $\left[\succ^*_i, \succ'_{-i}\right]$. By strategy-proofness of φ, j gets at least g' and no better than g at $[\succ^*_i, \succ_{-i}]$. Suppose j gets g' at $[\succ^*_i, \succ_{-i}]$. Then, by Maskin monotonicity of φ, we have $\varphi\left[\succ^*_i, \succ'_{-i}\right] = \varphi[\succ^*_i, \succ_{-i}]$. In particular, $\varphi\left[\succ^*_i, \succ'_{-i}\right](i) = \varphi[\succ^*_i, \succ_{-i}](i) = h$, contradicting that $\varphi\left[\succ^*_i, \succ'_{-i}\right](i) = \varphi[\succ'](i) \neq h$. Therefore, $\varphi[\succ^*_i, \succ_{-i}](j) = g$. QED

Claim 3. $\varphi[\succ^*_k, \succ_{-k}](k) = \varphi\left[\succ^*_k, \succ'_{-k}\right](k) = g$

Proof of Claim 3. By strategy-proofness of φ, since k gets h at $\succ$, k cannot get e and he gets at least h at $[\succ^*_k, \succ_{-k}]$. Thus, k gets h or g at $[\succ^*_k, \succ_{-k}]$. Everybody else ranks h over g. Thus by Pareto efficiency of φ, k should get g at $[\succ^*_k, \succ_{-k}]$.

By Claim 1, k gets h at $\succ'$, and a symmetric argument to the one above shows that k gets g at $\left[\succ^*_k, \succ'_{-k}\right]$. QED

Claim 4. $\varphi\left[\succ^*_k, \succ'_{-k}\right] = \varphi[\succ^*_k, \succ_{-k}]$.

Proof of Claim 4. These two profiles only differ in preferences of agent j who ranks g above g' at $\succ_j$ and the other way at $\succ'_j$. By Claim 3, j does not get g under $\succ_j$. Thus, Maskin monotonicity of φ implies $\varphi\left[\succ^*_k, \succ'_{-k}\right] = \varphi[\succ^*_k, \succ_{-k}]$. QED

Claim 5. If $\succ_i \in \langle e, h, g, ... \rangle$ then $\{\varphi[\succ^*_k, \succ_{-k}](i), \varphi[\succ^*_k, \succ_{-k}](i')\} = \{e, h\}$.

Proof of Claim 5. By Claim 3, agent k envies agent i at $\succ$. Thus, by Corollary 2, agent i gets at least $h = \varphi[\succ](k)$ at $[\succ_k^*, \succ_{-k}]$. Hence $\varphi[\succ_k^*, \succ_{-k}](i) \in \{e, h\}$. By Claim 1, $h = \varphi[\succ'](k)$ and the symmetric argument shows that $\varphi\left[\succ_k^*, \succ'_{-k}\right](i') \in \{e, h\}$. Furthermore, Claim 4 implies that $\varphi[\succ_k^*, \succ_{-k}](i') = \varphi\left[\succ_k^*, \succ'_{-k}\right](i')$. Thus $\varphi[\succ_k^*, \succ_{-k}](i)$ and $\varphi[\succ_k^*, \succ_{-k}](i')$ are different and both belong to $\{e, h\}$. QED

Claim 6. $\varphi[\succ_k^*, \succ_{-k}](i) = e$ and $\varphi[\succ_k^*, \succ_{-k}](i') = h$.

Proof of Claim 6. If not then Claim 5 implies that

$$\varphi[\succ_k^*, \succ_{-k}](i) = h \text{ and } \varphi[\succ_k^*, \succ_{-k}](i') = e.$$

By Claim 1 and Maskin monotonicity we can assume that

$$\succ_k \in \langle e, h, g, ...\rangle .$$

By Maskin monotonicity,

$$\varphi[\succ_k^*, \succ_{-k}] = \varphi[\succ_k^*, \succ_i^*, \succ_{-k,i}]$$

where $\succ_i^* \in \langle h, e, g, ...\rangle$. By the above equivalence and Claim 3,

$$\varphi[\succ_k^*, \succ_i^*, \succ_{-k,i}](k) = g.$$

By strategy-proofness of φ, k gets at least g and not e at $[\succ_i^*, \succ_{-i}]$. This contradicts Claim 2. QED

To complete the proof of the lemma, consider, $\succ_{i'}^* \in \langle e, h, g', ...\rangle$. By Maskin monotonicity

$$\varphi\left[\succ_{i'}^*, \succ'_{-i'}\right] = \varphi[\succ'] .$$

Furthermore, at $[\succ_{i'}^*, \succ_{-i'}]$, i would still get e and k would still get h. If j would still get g then we could assume that $\succ_{i'} = \succ_{i'}^*$ and the situation would be fully symmetric, contrary to Claims 4 and 6. Hence, j does not get g at $[\succ_{i'}^*, \succ_{-i'}]$. By strategy-proofness, j gets g' at $[\succ_{i'}^*, \succ_{-i'}]$. Thus, j is indifferent between submitting $\succ'_j$ and $\succ_j$ if other agents submit $[\succ_{i'}^*, \succ_{-i',j}]$. This contradicts non-bossiness. **QED**

Lemma 7 ***(Existence of a broker* for any b-house*)*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. If e is a b-house* at σ, then there exists an agent $k \in I - I_\sigma$ who is the broker* of e at σ.*

Proof of Lemma 7. We start with the following preparatory,
Claim 1. Suppose $h, h' \in (H - H_\sigma) - \{e\}$, $h \neq h'$, $\succ \in \mathbf{P}[\sigma, e, h]$, $\succ' \in \mathbf{P}[\sigma, e, h']$, $\succ'_\ell \in \langle e, h', h, ...\rangle$ for $\ell \in I - I_\sigma$, and $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$. Then $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Proof of Claim 1. By way of contradiction, suppose that $k = \varphi[\succ]^{-1}(h)$ is different from $k' = \varphi[\succ']^{-1}(h')$. Denote, $i = \varphi[\succ]^{-1}(e)$ and $i' = \varphi[\succ']^{-1}(e)$. We can assume that the ranking of houses other than h and h' in profiles $\succ$ and $\succ'$ is the same except for preferences of a single agent $j \in I - I_\sigma$. By Claim 1 of the proof of Lemma **6**, and by strategy-proofness, we can assume that $\succ_{k'}$ equals $\succ'_{k'}$ except for the relative ranking of h and h'. In particular, $\succ_{k'} \in \langle e, h, h', ...\rangle$. By Maskin monotonicity,

$$\varphi[\succ'_{k'}, \succ_{-k'}] = \varphi[\succ].$$

For agents $\ell \in I - I_\sigma - \{k'\}$, let $\succ^*_\ell$ equal $\succ'_\ell$ except for the relative ranking of h and h'. By Maskin monotonicity,

$$\varphi\left[\succ'_{k'}, \succ^*_{-k'}\right] = \varphi[\succ'].$$

By Lemma 6,

$$\varphi\left[\succ_{k'}, \succ^*_{-k'}\right]^{-1}(e) = \varphi[\succ]^{-1}(e) = i.$$

By strategy-proofness and non-bossiness φ (applied to agent k'), $\varphi\left[\succ_{k'}, \succ^*_{-k'}\right](k') = h$. Thus, Corollary 2 implies that

$$h = \varphi\left[\succ'_{k'}, \succ^*_{-k'}\right](i).$$

Now, notice that

$$\left[\succ'_{k'}, \succ^*_{-j,k'}\right] = \left[\succ'_{k'}, \succ_{-j,k'}\right].$$

Thus, agent j envies i at $\left[\succ'_{k'}, \succ^*_{-k'}\right]$ and can improve i allocation by submitting $\succ_j$ instead of $\succ^*_j$, which contradicts Lemma 3. QED

Claim 2. Suppose $h, h' \in (H - H_\sigma) - \{e\}$, $h \neq h'$, $\succ \in \mathbf{P}[\sigma, e, h]$, $\succ' \in \mathbf{P}[\sigma, e, h']$, $\succ'_\ell \in \langle e, h', h, ...\rangle$ for $\ell \in I - I_\sigma$. Then $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Proof of Claim 2. If $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$, then we are done by Claim 1. Therefore, assume that $\varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e)$. Because e is a b-house* at σ, there exists some $h'' \in (H - H_\sigma) - \{e\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, e, h'']$,

$$\varphi[\succ'']^{-1}(e) \neq \varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e).$$

By Lemma 6, $h'' \neq h$, and by the same lemma we can assume that $\succ'_\ell \in \langle e, h'', h, ...\rangle$ for $\ell \in I - I_\sigma$. By Claim 1, $\varphi[\succ'']^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ'']^{-1}(h'') = \varphi[\succ']^{-1}(h')$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$. QED

Claim 3. Suppose $h \in (H - H_\sigma) - \{e\}$, $\succ, \succ' \in \mathbf{P}[\sigma, e, h]$. Then $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Proof of Claim 3. By Lemma 6, $\varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e)$. Because e is a b-house* at σ, there exists some $h'' \in (H - H_\sigma) - \{e\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, e, h'']$,

$$\varphi[\succ'']^{-1}(e) \neq \varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e).$$

By Lemma 6, $h'' \neq h$, and by the same lemma we can assume that $\succ'_\ell \in \langle e, h'', h, ... \rangle$ for $\ell \in I - I_\sigma$. By Claim 1, $\varphi[\succ'']^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ'']^{-1}(h'') = \varphi[\succ']^{-1}(h')$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$. QED

Because of Claim 3, to finish that proof of the lemma, it is enough to show that for all $h, h' \in (H - H_\sigma) - \{e\}$, $h \neq h'$, and all $\succ \in \mathbf{P}[\sigma, e, h]$, $\succ' \in \mathbf{P}[\sigma, e, h']$,

$$\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h').$$

Consider $\succ^* \in \mathbf{P}[\sigma, e, h]$ and such that $\succ^*_\ell \in \langle e, h, h', ... \rangle$ for $\ell \in I - I_\sigma$ and $\succ^{*\prime} \in \mathbf{P}[\sigma, e, h']$ and such that $\succ^{*\prime}_\ell \in \langle e, h', h, ... \rangle$ for $\ell \in I - I_\sigma$. By Claim 2,

$$\varphi[\succ]^{-1}(h) = \varphi[\succ^{*\prime}]^{-1}(h) = \varphi[\succ^*]^{-1}(h) = \varphi[\succ']^{-1}(h').$$

QED

Lemma 8 ***(Relationship between brokerage* and strong ownership*).*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. If agent k is a broker* of house e at σ, and $\succ'' \in \mathbf{P}^*[\sigma, e]$, then agent $\varphi[\succ'']^{-1}(e)$ is the strong owner* of house $\varphi[\succ''](k)$ at σ.*

Proof of Lemma 8. Let $\succ'' \in \mathbf{P}^*[\sigma, e]$ and $h = \varphi[\succ''](k)$. Because k is a broker* at σ, Lemma 7 implies that house h is agent k second choice. Since $\succ'' \in \mathbf{P}^*[\sigma, e]$, house h is the second choice of all $I - I_\sigma$ agents at $\succ''$, and thus

$$\succ'' \in \mathbf{P}[\sigma, e, h].$$

There exists an agent $i \in (I - I_\sigma) - \{k\}$ such that $\varphi[\succ'']^{-1}(e) = i$. By Lemma 6, for any $\succ \in \mathbf{P}[\sigma, e, h]$ agent i gets e at $\succ$. We are to show that i is the strong owner* of h at σ.

Claim 1. If $\succ \in \mathbf{P}[\sigma, e, h]$ then $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$. Indeed, it the first claim follows from Lemma 6, and the second from Lemma 7.

Step 1. Let preference profile $\succ$ be such that $\succ_{i'} = \succ''_{i'}$ for all $i' \in \{k, i\} \cup I_\sigma$, and all houses in $H - H_\sigma$ are ranked above the houses in H_σ by $i' \in I - I_\sigma$. By Claim 1, and the Maskin monotonicity of φ, $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$.

Step 2. Let $\succ^*_i \in \langle h, e, ... \rangle$. By strategy-proofness of φ, since $\varphi[\succ](i) = e$, agent i gets at least e at $[\succ^*_i, \succ_{-i}]$, and since all other agents in $I - I_\sigma$ prefer e over h, Pareto efficiency of φ implies that $\varphi[\succ^*_i, \succ_{-i}](i) = h$.

Step 3. Let $\succ^*_k \in \langle h, e, ... \rangle$. Since $\varphi[\succ](k) = h$, profile $[\succ^*_k, \succ_{-k}]$ is a monotonic extension of $\succ$ and by Maskin monotonicity of φ, we have $\varphi[\succ^*_k, \succ_{-k}] = \varphi[\succ]$.

Step 4. Agent i gets h at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. By Step 3, $\varphi\left[\succ^*_k, \succ_{-k}\right](i) = \varphi\left[\succ\right](i) = e$, and, by strategy-proofness of φ, i gets at least e at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. Thus, if i does not get h at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$ then one of the following two cases would have to obtain.

Case 1. An agent $j \neq i, k$ gets h at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. Then i gets e, and k gets some house worse than e. But then jointly i and k can report $\succ_{\{i,k\}}$ instead of $\succ^*_{\{i,k\}}$, and they would jointly improve, i.e., $\varphi\left[\succ\right](i) = e = \varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i)$ and $\varphi\left[\succ\right](k) = h \succ^*_k \varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](k)$, contradicting φ is coalitionally strategy-proof.

Case 2. Agent k gets h at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. By strategy-proofness of φ, agent k should at least get h at $\left[\succ^*_i, \succ_{-i}\right]$. But we know by Step 2 that $\varphi\left[\succ^*_i, \succ_{-i}\right](i) = h$, thus we should have $\varphi\left[\succ^*_i, \succ_{-i}\right](k) = e$. Then by Maskin monotonicity of φ, we have $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = \varphi\left[\succ^*_i, \succ_{-i}\right](i)$ and equals h (by Step 2). A contradiction that proves the claim of Step 5.

Step 5. If $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = h$ then $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](k) \neq e$. For an indirect argument, suppose that $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = h$ and $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](k) = e$. Then, $\varphi\left[\succ^*_i, \succ_{-i}\right](k) = e$ by strategy-proofness of φ. Since e is a σ-b-house*, there exists some house $g \neq e, h$, preference profile $\succ' \in \mathbf{P}\left[\sigma, e, g\right]$ such that $\varphi\left[\succ'\right]^{-1}(e) = j$ for some agent $j \neq i, k$. By Lemma 6, we may assume that each agent $i' \in I - I_\sigma$ ranks houses other than $\{h, g\}$ in the same way at $\succ'_{i'}$ and $\succ_{i'}$ and that $\succ'_{i'} \in \langle e, g, h, ...\rangle$. Since k is the σ-broker* of e, we have $\varphi\left[\succ'\right](k) = g$. By Maskin monotonicity,

$$\varphi\left[\succ'\right] = \varphi\left[\succ'_{k,i}, \succ_{-k,i}\right].$$

Now i gets a house weakly worse than h at $\left[\succ'_{k,i}, \succ_{-k,i}\right]$. However, if i and k manipulated and submitted $\succ^*_{i,k}$ instead of $\succ'_{i,k}$, they would get h and e respectively at $\left[\succ^*_{k,i}, \succ_{-k,i}\right]$. Both agents weakly improve, while k strictly improves. This contradicts the fact that φ is coalitionally strategy-proof.

Now, Steps 4 and 5 imply that $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = h$ and $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](k) \neq e$. By Maskin monotonicity, we can drop the ranking of e in $\succ^*_i$ and $\succ^*_k$, and yet, the outcome of φ will not change. Recall that $\succ_{-i,k}$ was an arbitrary profile in which all houses in $H - H_\sigma$ are ranked above the houses in H_σ by $i' \in I - I_\sigma - \{i, k\}$. Thus, i gets h at all profiles of $\mathbf{P}\left[\sigma, h\right]$. **QED**

We state the following corollary to Lemma 8.

Corollary 3 *Let $\sigma \in \mathcal{S} - \mathcal{M}$, and (k, e) be a b-pair* at σ, and (i, h) be an o-pair* at σ. Then, we have the following:*

- *(i, h) is a strong o-pair* at σ.*
- *If $\succ$ is a preference profile such that $\succ_i \in \langle e, ...\rangle$ and $\succ_k \in \langle e, h, ...\rangle$, then $\varphi\left[\succ\right](i) = e$, and $\varphi\left[\succ\right](k) = h$.*

Lemma 9 *(**Uniqueness of a broker***). Let $\sigma \in \mathcal{S} - \mathcal{M}$. If k is a broker* of e at σ, then no other agent $j \neq k$ is a broker* of e at σ.*

Proof of Lemma 9. Suppose k is a broker* of e at σ. Pick $\succ \in \mathbf{P}^* [\sigma, e]$. By definition, $\varphi [\succ] (k) = h$ where h is the second choice house of every agent in $I - I_\sigma$. Since no other agent in $I - I_\sigma$ gets his second choice, there is no other broker* of e. **QED**

Lemma 10 *(**Uniqueness of a b-house***). Let $\sigma \in \mathcal{S} - \mathcal{M}$. If e is a b-house* at σ, then no other house is a b-house* at σ (and all other unmatched houses are strong o-houses*).*

Proof of Lemma 10. Let e be a b-house* at σ. By Lemma 7, there is a broker* of e at σ, let us denote him k. Consider a house $h \in I - I_\sigma - \{e\}$. By Lemma 6, there is an agent i who gets e at any profile in $\mathbf{P} [\sigma, e, h]$. By Lemma 8, i is the strong owner* of h. **QED**

Lemma 11 *(**Broker* does not own***). Let $\sigma \in \mathcal{S} - \mathcal{M}$. If agent k is a broker* of house e at σ, then he cannot own* any houses at σ.*

Proof of Lemma 11. Suppose that k owns* a house $h \neq e$ at σ. By Lemma 6, there exists some agent $i \neq k$ who gets e at all profiles in $\mathbf{P} [\sigma, e, h]$. By Lemma 8, (i, h) is a strong o-pair* at σ. By definition, i gets h at all $\succ \in \mathbf{P}^* [\sigma, h]$, contradicting that k owns* h. **QED**

B.2 "Intertemporal" Lemmas (Persistence of Ownership and Brokerage)

Lemma 12 *(**Persistence of a strong o-pair***). Let (i, h) be a strong o-pair* at some $\sigma \in \mathcal{S} - \mathcal{M}$. If $\sigma' \supset \sigma$, and i and h are unmatched at σ', then (i, h) is a strong o-pair* at σ'.*

Proof of Lemma 12. Imagine to the contrary that i gets h at all $\succ \in \mathbf{P} [\sigma, h]$, but there is some $\succ' \in \mathbf{P} [\sigma', h]$ such that some agent $j \in I - I_{\sigma'}$, $j \neq i$, gets h at $\succ'$. Take $\succ \in \mathbf{P} [\sigma, h]$ such that

- for each agent $k \notin I_{\sigma'} - I_\sigma$, $\succ_k = \succ'_k$, and
- each agent $k \in I_{\sigma'} - I_\sigma$ ranks $\sigma' (k)$ as his second choice (just behind h) in $\succ_k$.

The only difference between the profiles $\succ'$ and $\succ$ are the preferences of the agents in $I_{\sigma'} - I_\sigma$. Moreover, each $k \in I_{\sigma'} - I_\sigma$ is indifferent between $\succ'$ and $\succ$ because:

- at $\succ'$ agent k gets $\sigma' (k)$ by Lemma 5,
- at $\succ$ agent k gets $\sigma' (k)$ by Pareto efficiency of φ and the fact that $\varphi [\succ] (i) = h$.

Thus, agents $I_{\sigma'} - I_\sigma$ are indifferent between $\succ$ to $\succ'$ while agent j is strictly better off at $\succ'$. This contradicts the fact that φ is coalitionally strategy-proof. **QED**

Lemma 13 ***(Limited persistence of b-pairs*).*** *Let $\sigma, \sigma' \in \mathcal{S} - \mathcal{M}$ and $\sigma' \supset \sigma$. Suppose that agent k is the σ-broker* of house e, agent i is the strong σ-owner* of house h, and $i' \neq i$ is the strong σ-owner* of h'. If k, i, i', e, h, h' are unmatched at σ', then k is the σ'-broker* of e.*

Proof of Lemma 13. First notice that i gets e at $\mathbf{P}[\sigma, e, h]$ and i' gets e at $\mathbf{P}[\sigma, e, h']$ (and k gets h or h' respectively). Take $\succ \in \mathbf{P}[\sigma, e, h]$ and $\succ' \in \mathbf{P}[\sigma, e, h']$ such that each agent $j \in I_{\sigma'} - I_\sigma$ have $\sigma'(j)$ as his third choice and each agent $j \in I - I_{\sigma'}$ ranks each house unmatched at σ' above all houses matched at σ'. Let profile $\succ^*$ be obtained from $\succ$ by moving $\sigma'(j)$ for any $j \in I_{\sigma'} - I_\sigma$ up to be the first choice of j. Let $\succ'^*$ be obtained analogously from $\succ'$. By Maskin monotonicity, $\varphi[\succ^*]^{-1}(e) = i \neq i' = \varphi[\succ'^*]^{-1}(e)$. Hence, e is a b-house* at σ'.

If now k is not the broker* of e at σ', then there exists some $g \neq e$ and $\succ' \in \mathbf{P}[\sigma', e, g]$ such that $k' = \varphi[\succ']^{-1}(g) \neq k$. Let $\succ \in \mathbf{P}[\sigma, e, g]$, and $j = \varphi[\succ]^{-1}(e) \in I - I_\sigma$.

Step 1. $j \in I_{\sigma'} - I_\sigma$ and $g \neq h, h'$. First suppose $j \in I - I_{\sigma'}$. We have $\varphi[\succ](k) = g$, since (k, e) is a σ-b-pair*. Suppose each agent j' in $I_{\sigma'} - I_\sigma$ list $\sigma(j')$ as his third choice at $\succ$ and each agent in $I - I_{\sigma'}$ list houses in $H_{\sigma'}$ lower than the other houses at $\succ$ (this is without loss of generality, since by Lemma 6, j still gets e). Then, by Maskin monotonicity of φ, we have $\varphi[\succ'] = \varphi[\succ]$, contradicting $\varphi[\succ](k) = g \neq \varphi[\succ](k')$. We showed that $j \in I_{\sigma'} - I_\sigma$. Moreover, since $i, i' \in I - I_{\sigma'}$, $j \neq i, i'$. Since i does not get e or g at $\succ$, and since (i, h) is σ-o-pair*, i can at least get h at $\succ$, implying that $h \neq g$. By a similar argument, $h' \neq g$.

Step 2. Since (k, e) is σ-b-pair*, by Corollary 3, for any profile $\succ'' \in \mathbf{P}[\sigma]$ with $\succ''_i \in \langle e, ... \rangle$ and $\succ''_k \in \langle e, h, ... \rangle$, we have $\varphi[\succ''](i) = e$ and $\varphi[\succ''](k) = h$.

Step 3. Step 2 and the intratemporal lemmas imply that either (i, e) is a strong o-pair* at σ' or (k, e) is a b-pair* at σ'. However, we arbitrarily chose i, and the same conditions should be satisfied by i', e, k with $i' \neq i$, as well. Hence if (i, e) is a strong o-pair* then (k, e) is not a b-pair* at σ', and (i', e) is a strong o-pair* at σ', contradicting the fact that a house can only be owned* by a single agent. Thus, (k, e) is a b-pair* at σ'. **QED**

Lemma 14 ***(Broker*-to-heir* transition)*** *Let $\sigma, \sigma' \in \mathcal{S} - \mathcal{M}$ and $\sigma' = \sigma \cup \{j, g\} \supsetneq \sigma$. Suppose that $k \neq j$ brokers* $e \neq g$ at σ, and i owns* h both at σ and σ'. If $\succ \in \mathbf{P}[\sigma']$, and $\varphi[\succ](i) = e$, then $\varphi[\succ](k) \succsim_k h$.*

Proof of Lemma 14. By Corollary 3, for any profile $\succ \in \mathbf{P}[\sigma]$ with $\succ_i \in \langle e, ... \rangle$, and $\succ_k \in \langle e, h, ... \rangle$ we have $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$. Thus, by coalitional strategy-proofness of φ, the lemma should hold. **QED**

B.3 Proof of Theorem 4

Proof of Theorem 4. Let φ be a coalitionally strategy-proof and Pareto-efficient mechanism. We will construct a compatible control rights structure (c, b). Fix $\sigma \in \mathcal{S} - \mathcal{M}$. For any $h \in H - H_\sigma$, two cases are possible:
Case 1. $\varphi[\succ]^{-1}(h)$ is constant for all $\succ \in \mathbf{P}^*[\sigma, h]$. Then $\left(\varphi[\succ]^{-1}(h), h\right)$ is an o-pair* at σ. We set

$$c_h(\sigma) = \varphi[\succ]^{-1}(h).$$

Case 2. There exist $\succ, \succ' \in \mathbf{P}^*[\sigma, h]$ such that $\varphi[\succ]^{-1}(h) \neq \varphi[\succ']^{-1}(h)$. Then there exists $k \in I - I_\sigma$, such that (k, h) is a b-pair* (that is, for all $\succ^* \in \mathbf{P}^*[\sigma, h]$, $\varphi[\succ^*](k)$ is the second choice of agent k in $\succ^*(k)$). In this case, we set

$$c_h(\sigma) = k \text{ and}$$
$$b(\sigma) = h.$$

Additionally, if Case 2 does not hold for any house $h \in H - H_\sigma$, we set

$$b(\sigma) = \emptyset.$$

Lemmas 8-9 and Corollary 3 imply that (c, b) is well-defined. Lemmas 12- 14 show that the control right structure (c, b) is compatible, hence the TCBO mechanism $\psi^{c,b}$ is well-defined. More precisely, Lemma 12 shows that C1 holds. Lemma 11 shows that C2 holds. Lemmas 13 and 14 show that C3 holds. Furthermore, the intratemporal lemmas show that for any $\sigma \in \mathcal{S} - \mathcal{M}$, o-pairs* and b-pairs* of mechanism φ coincide with o-pairs and b-pairs of mechanism $\psi^{c,b}$.

To prove that $\varphi = \psi^{c,b}$, fix $\succ \in \mathcal{P}^{|I|}$. We will show that $\varphi[\succ] = \psi^{c,b}[\succ]$. Let I^r be the set of agents removed in round r of $\psi^{c,b}$. For each agent $i \in I^r$, there is a unique house that points to him and is removed in the same cycle as i; let us denote this house h_i. Let us construct the following preference profile $\succ^*$ by modifying $\succ$.

- If $\psi^{c,b}[\succ](i) = h_i$, then $\succ_i^* = \succ_i$.
- If $\psi^{c,b}[\succ](i) \neq h_i$, and either no b-house was removed in the same cycle as i or a b-house was assigned to i, then we construct $\succ_i^*$ from $\succ_i$ by moving h_i just after $\psi^{c,b}[\succ](i)$ (we do not change the ranking of other houses).
- If i is removed as owner and a b-house $e^r \neq \psi^{c,b}[\succ](i)$ was removed in the same cycle as i, then we construct $\succ_i^*$ from $\succ_i$ by moving e^r just after $\psi^{c,b}[\succ](i)$ and moving h_i just after e^r.
- If a broker k^r is removed in the cycle

  $$h_{i^1} \to i^1 \to h_{i^2} \to i^2 \to ...h_{i^n} \to i^n \to e^r \to k^r \to h_{i^1},$$

  then we construct $\succ_{k^r}^*$ from $\succ_{k^r}$ by moving h_{i^n} just below h_{i^1}.

Observe that $\psi^{c,b}[\succ^*] = \psi^{c,b}[\succ]$. Moreover, since

$$\left\{ h \in H : h \succeq_i \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} = \left\{ h \in H : h \succeq_i^* \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} \qquad \forall i \in I, \tag{1}$$

$\succ^*$ is a monotonic extension of $\succ$ at $\psi^{c,b}$ and $\succ$ is a monotonic extension of $\succ^*$ at $\psi^{c,b}$.

We will next prove that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r} I^s = I_{\sigma^r}, \qquad \forall r = 0, 1, 2, ... \tag{2}$$

by induction over r. The claim is trivially true for $r = 0$. Fix round $r \geq 1$ and let σ^{r-1} be the matching fixed before round r (in particular, $\sigma^0 = \varnothing$). For the inductive step, assume that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r-1} I^s = I_{\sigma^{r-1}} \tag{3}$$

for . We will prove that the same expression holds for agents in I^r using the following three claims.

Claim 1. $\varphi[\succ^*](i) \succeq_i^* h_i$ for any owner $i \in I^r$.

Proof of Claim 1. Let $\succ' \in \mathbf{P}^*[\sigma^{r-1}, h_i]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}} - \{h_i\}$ in $\succ'_j$ is the same as in $\succ^*_j$ for $j \in (I - I_{\sigma^{r-1}}) - \{i\}$, and let $\succ'' \in \mathbf{P}[\sigma^{r-1}]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ in $\succ''_j$ is the same as in $\succ^*_j$ for $j \in (I - I_{\sigma^{r-1}}) - \{i\}$.

If $i' \in I_{\sigma^{r-1}}$ then

$$\varphi[\succ'](i') = \varphi[\succ''](i') = \sigma^{r-1}(i') = \psi^{c,b}[\succ^*](i') = \varphi[\succ^*](i'),$$

by construction of $\mathbf{P}^*[\sigma^{r-1}, h_i], \mathbf{P}[\sigma^{r-1}]$, and σ^{r-1}, and by the inductive assumption. Since (i, h_i) is an o-pair at σ^{r-1} hence, it is an o-pair* by construction of (c, b). By definition of an o-pair*,

$$\varphi[\succ'](i) = h_i. \tag{4}$$

Thus, no agent $j \in (I - I_{\sigma^{r-1}}) - \{i\}$ gets a house in $\{h_i\} \cup H_{\sigma^{r-1}}$ under $\varphi[\succ']$.

By Maskin monotonicity,

$$\begin{aligned} \varphi[\succ^*] &= \varphi\left[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ^*_{I_{\sigma^{r-1}} \cup \{i\}}\right] \\ &= \varphi\left[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ'_{I_{\sigma^{r-1}}}, \succ^*_i\right], \end{aligned} \tag{5}$$

and

$$\varphi[\succ'] = \varphi\left[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ'_{I_{\sigma^{r-1}} \cup \{i\}}\right]. \tag{6}$$

By Equation 5, strategy-proofness of φ, and Equations 6 and 4, we have

$$\varphi\left[\succ^*\right](i)=\varphi\left[\succ''_{\left(I-I_{\sigma^{r-1}}\right)-\{i\}},\succ'_{I_{\sigma^{r-1}}},\succ^*_i\right](i)\ \ \succeq^*_i\ \varphi\left[\succ''_{\left(I-I_{\sigma^{r-1}}\right)-\{i\}},\succ'_{I_{\sigma^{r-1}}\cup\{i\}}\right](i)=\varphi\left[\succ'\right](i)=h_i.$$

QED

Claim 2. If $i\in I^r$ and no b-house was removed in the cycle of i, then $\varphi\left[\succ^*\right](i)=\psi^{c,b}\left[\succ^*\right](i)$.

Proof of Claim 2. The inductive assumption implies that all houses better than $\psi^{c,b}\left[\succ^*\right](i)$ are already given to other agents, hence

$$\psi^{c,b}\left[\succ^*\right](i)\succeq^*_i\varphi\left[\succ^*\right](i).$$

For an indirect argument, suppose $\varphi\left[\succ^*\right](i)\neq\psi^{c,b}\left[\succ^*\right](i)$. Then, Claim 1 and the construction of $\succ^*$ imply that

$$\varphi\left[\succ^*\right](i)=h_i.$$

Let

$$h_i\rightarrow i\rightarrow h_{i^2}\rightarrow i^2\rightarrow...\rightarrow h_{i^n}\rightarrow i^n\rightarrow h_i$$

be the cycle in which i is removed under $\psi^{c,b}\left[\succ^*\right]$. From

$$\varphi\left[\succ^*\right](i)=h_i=\psi^{c,b}\left[\succ^*\right](i^n),$$

we conclude that $\varphi\left[\succ^*\right](i^n)\neq\psi^{c,b}\left[\succ^*\right](i^n)$, and Claim 1 and the construction of $\succ^*$ imply that

$$\varphi\left[\succ^*\right](i^n)=h_{i^n}=\psi^{c,b}\left[\succ^*\right]\left(i^{n-1}\right).$$

As we continue iteratively, we obtain that

$$\varphi\left[\succ^*\right](j)=h_j$$

for all $j\in\{i,i^2,...,i^n\}$. Hence, the matching obtained by assigning $\psi^{c,b}\left[\succ^*\right](j)$ to each agent $j\in\{i,i^2,...,i^n\}$ and $\varphi\left[\succ^*\right](j)$ to each agent $j\in I-\{i,i^2,...,i^n\}$ Pareto-dominates $\varphi\left[\succ^*\right]$ at $\succ^*$, contradicting that $\varphi\left[\succ^*\right]$ is Pareto-efficient. QED

Claim 3. If $i\in I^r$ and a b-house was removed in the cycle of i, then $\varphi\left[\succ^*\right](i)=\psi^{c,b}\left[\succ^*\right](i)$.

Proof of Claim 3. Let e^r be the σ^{r-1}-b-house and k^r be the σ^{r-1}-broker removed in the cycle

$$h_{i^1}\rightarrow i^1\rightarrow h_{i^2}\rightarrow i^2\rightarrow...h_{i^{n+1}}\rightarrow i^{n+1}\rightarrow h_{i^1}$$

where $i^{n+1}=k^r$ and $h_{i^{n+1}}=e^r$. For any $i^\ell\in\{i^1,...,i^n\}$, by the inductive assumption all houses better than $h_{i^{\ell+1}}$ are already given to other agents, hence Claim 1 implies that

$$\varphi\left[\succ^*\right]\left(i^\ell\right)\in\left\{h_{i^\ell},e^r,\psi^{c,b}\left[\succ^*\right]\left(i^\ell\right)\right\}=\left\{h_{i^\ell},h_{i^{n+1}},h_{i^{\ell+1}}\right\}. \qquad (7)$$

To show that

$$\varphi\left[\succ^*\right](k^r) = h_{i^1} = \psi^{c,b}\left[\succ^*\right](k^r), \tag{8}$$

assume otherwise. Then, $\varphi\left[\succ^*\right](k^r) = \varphi\left[\succ^*\right](i^{n+1}) = h_{i^{n+1}}$. By Equation 7, $\varphi\left[\succ^*\right](i^1) = h_{i^1}$. Let $\succ' \in \mathbf{P}\left[\sigma^{r-1}\right]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ in $\succ'_j$ is the same as in $\succ^*_j$ for $j \in I$ except that agents k^r and i^1 moved e^r to be their first choice and h_{i^1} to be their second. By Maskin monotonicity,

$$\varphi\left[\succ'\right] = \varphi\left[\succ^*\right].$$

By construction of (c, b), k^r brokers* e^r and i^1 owns* h_{i^1} at σ^{r-1}, and by Corollary 3, $\varphi\left[\succ'\right](k^r) = h_{i^1} \neq h_{i^{n+1}}$. Thus the claim of the paragraph is proved.

Let us now show that also

$$\varphi\left[\succ^*\right]\left(i^\ell\right) \in \{e^r, h_{i^{\ell+1}}\}$$

for $\ell = 1, ..., n$. For $\ell = 1$ the claim follows from Equations 7 and 8. Consider $\ell = 2$. If $\varphi\left[\succ^*\right](i^2) = h_{i^2}$ then it must be that $\varphi\left[\succ^*\right](i^1) = e^r$. But then i^1 and i^2 will swap e^r and h_{i^2} leading to a Pareto improvement, which is impossible as φ is Pareto-efficient. Thus, we showed that $\varphi\left[\succ^*\right](i^2) \in \{e^r, h_{i^3}\}$. Induction concludes the argument.

Since $h_{i^{n+1}} = e^r$, we know that

$$\varphi\left[\succ^*\right](i^n) = e^r = h_{i^{n+1}},$$

and iteratively

$$\varphi\left[\succ^*\right]\left(i^\ell\right) = h_{i^{\ell+1}} = \psi^{c,b}\left[\succ^*\right]\left(i^\ell\right).$$

for $\ell = 1, ..., n$. This identity and Equation 8 prove the claim. QED

Let σ^r be the matching fixed after Round r. By the inductive assumption, and by Claims 2 and 3, $\varphi\left[\succ^*\right](i) = \psi^{c,b}\left[\succ^*\right](i)$ for all $i \in I_{\sigma^r}$. This completes the induction, and the proof of Statement 2.

The theorem follows from , $\psi^{c,b}\left[\succ\right] = \psi^{c,b}\left[\succ^*\right]$, $\psi^{c,b}\left[\succ^*\right] = \varphi\left[\succ^*\right]$, and $\varphi\left[\succ^*\right] = \varphi\left[\succ\right]$. The first of these observations is straightforward through the construction of $\succ^*$. The second one follows from Statement 2. The third one follows from Maskin monotonicity of φ, because $\psi^{c,b}\left[\succ^*\right] = \varphi\left[\succ^*\right]$ and Statement 1 imply that

$$\{h \in H : h \succeq_i \varphi\left[\succ^*\right](i)\} = \{h \in H : h \succeq^*_i \varphi\left[\succ^*\right](i)\} \text{ for } i \in I.$$

QED

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