# Decomposing Random Mechanisms*

Marek Pycia †
*UCLA*

M. Utku Ünver ‡
*Boston College*

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**Abstract**

Random mechanisms have been used in real-life situations for reasons such as fairness. Voting and matching are two examples of such situations. We investigate whether desirable properties of a random mechanism survive decomposition of the mechanism as a lottery over deterministic mechanisms that also hold such properties. To this end, we represent properties of mechanisms—such as ordinal strategy-proofness or individual rationality—using linear constraints. Using the theory of totally unimodular matrices from combinatorial integer programming, we show that total unimodularity is a sufficient condition for the decomposability of linear constraints on random mechanisms. As two illustrative examples, we show that individual rationality is totally unimodular in general, and that strategy-proofness is totally unimodular in some individual choice models. We also introduce a second, more constructive approach, to decomposition problems, and prove that feasibility, strategy-proofness, and unanimity, with and without anonymity, are decomposable on non-dictatorial single-peaked voting domains. As importantly, we also establish that strategy-proofness is not decomposable in some natural problems.

**Keywords:** Random mechanisms, ordinal mechanisms, total unimodularity, single-peaked preferences, voting, individual rationality, strategy-proofness, unanimity, anonymity, generalized median voter rules, universal truthfulness.

**JEL Classification:** C60, D71, D72

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†UCLA Department of Economics, 8283 Bunche Hall, Los Angeles CA 90095. E-mail: pycia@econ.ucla.edu.

‡Boston College, Department of Economics, Chestnut Hill, MA, 02467. Email: unver@bc.edu.

# 1 Introduction

Random mechanisms are frequently used in sustaining fairness among market participants. For example, admission to public schools through school choice in the US (cf. Abdulkadiroğlu and Sönmez, 2003) is administered in many districts through centralized random mechanisms that use random tie-breakers. Some voting and social-choice environments also use random mechanisms. Jury selection, draft lotteries, and ballot positioning are further examples (cf. Fishburn, 1984). Other examples include voting in olympic figure skating competitions and the election method of military leaders (known as doges) in Venice (used for more than 500 years; cf. Lines, 1986). Some random mechanisms are designed directly to use a lottery over predetermined deterministic mechanisms, as in school choice. Another approach in random mechanism design uses probabilistic assignment over outcomes for each situation rather than deterministic mechanisms in the support of the random mechanism. Competitive equilibrium from equal incomes of Hylland and Zeckhauser (1979), the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) for object allocation, and maximal lottery methods (cf. Kreweras, 1965; Fishburn, 1984) for voting are some examples of this approach.

Random mechanisms correspond to the full range of possible mechanisms. From the point of view of mechanism design, they cannot be neglected in the search for the best mechanism to implement a desired goal. On the other hand, many market design situations require transparency of the mechanism. Randomness of a mechanism is often a source of additional complexity in explaining and educating the agents who will participate in its implementation. Although simple tie-breakers can easily be explained to the participants in certain situations (e.g., in school choice), more complex random mechanism implementation often hinges on the condition that we can implement a deterministic mechanism to represent the random mechanism. For this reason, the market designer may want to resolve the uncertainty regarding the mechanism as soon as possible, before the participants' private information is collected. Thus, the representability of a random mechanism as a randomization over deterministic mechanisms that also have the same properties could be crucial to the success of the design.

When a property is transferable through decomposition, it holds both ex ante, i.e., before the uncertainty regarding the mechanism is resolved, and ex post, i.e., after this uncertainty is resolved. In this case, the mechanism is more robust and is not affected by the market participants' access to information regarding the resolution of the uncertainty in the mechanism. For example, if dominant-strategy incentive-compatibility (or strategy-proofness) is decomposable, then it is best for an agent to reveal his preferences truthfully regardless of if all he knows is that a stochastically strategy-proof mechanism will be implemented or if he knows exactly, after the lottery is resolved, which strategy-proof deterministic mechanism will be implemented. If such a decomposition goes through, this deterministic mechanism, in many cases, can be explained more transparently to the participants. [1]

The goal of this paper is to narrow the gap between our understanding of random and deterministic mechanisms in ordinal environments. Although we have a good understanding of which properties of deterministic mechanisms are preserved when we randomize over deterministic mechanisms, the other direction remains quite unclear. Exploring the possibility or impossibility of decomposition of a property shows whether or not, without loss of generality, we can focus on lotteries over deterministic mechanisms in mechanism design.

We adopt two approaches in determining the decomposability of properties of random mechanisms. We start with formulating a simple sufficient condition and then use a constructive approach for more complex properties where this first approach is inconclusive.

First, we reformulate a useful approach to mechanism design, which has been used in combinatorial integer programming in various applications. We show how to analyze which properties of a random mechanism are decomposable by employing *totally unimodular* (*TUM*) *decomposition* (cf. Theorem 1). In this way, we contribute to the growing literature on new approaches to mechanism design using linear programming tools, which have recently found their way to mainstream economics (see Vohra, 2011). Using these methods, we show that every individually rational random mechanism is a lottery over individually rational deterministic mechanisms in a variety of environments including object allocation, social choice, and matching (cf. Theorem 2). Strategy-proofness with and without individual rationality constraints are also TUM in certain models. We give an example of an individual choice model where strategy-proofness is TUM and hence decomposable (cf. Theorem 3).[2,3]

Surprisingly, we find a counter-example that even with a single agent, in the universal house allocation or voting domains, strategy-proofness is not decomposable, and hence not TUM (cf. Theorem 4). On the other hand, together with other properties, strategy-proofness

[1] In algorithmic game theory, the computer science literature that deals with game theory and mechanism design, decomposability of a property has also attracted special attention. The literature refers to decomposability of a property as *universality* (cf. Nisan and Ronen, 1999). For example, universal strategy-proofness (or truthfulness, as sometimes referred to in the computer science literature) is inspected in a recent paper by Krysta et al. (2014) to find a matching that matches as many agents as possible without sacrificing universal strategy-proofness in the house allocation problem. They find an upper approximation bound for this problem.

[2] A deterministic mechanism is individually rational if its outcome is preferred by agents to their outside options. A random mechanism is individually rational if its outcome first-order stochastically dominates agents' outside options.

[3] Observe that there could be other ways of proving that these properties are decomposable in the aforementioned domains. Some of these proofs are simpler and they may not need the TUM property. However, our theorems are stronger than just showing these properties are decomposable, as we prove that they are TUM (a sufficient but not necessary condition for decomposability).

can still be decomposable on these domains.

Moreover, TUM decomposability is sometimes too strong. Even though a property is not TUM, it could still be decomposable. For example, it is straightforward to show that on the single-peaked voting domain (and hence on the universal domain), strategy-proofness, unanimity, and feasibility taken together are not TUM.[4,5] Despite this fact, we prove that they are decomposable (cf. Theorem 5). In proving this result, we employ a constructive approach, which requires the knowledge of the characterization of deterministic mechanisms that carry the same properties as the random mechanism. Using this information, we construct a lottery over deterministic mechanisms with the required properties that induces a given random mechanism. Moreover, we prove that strategy-proofness is decomposable for tops-only mechanisms (i.e., when the mechanism outcome relies only on the reported top choices of the agents) on a single-peaked voting domain and unanimity is not needed for this result as an additional property (cf. Theorem 6).

As a corollary to the proof of decomposability of strategy-proofness and unanimity on a single-peaked voting domain, we also establish that anonymity, unanimity, strategy-proofness, and feasibility are jointly decomposable (cf. Theorem 7).[6]

A forerunner to our work, Gibbard (1977) studied the decomposition of strategy-proofness in voting when all strict preference rankings are admissible, i.e., on the universal social-choice domain. In this model, he showed that any unanimous and strategy-proof random mechanism is a randomization over unanimous and strategy-proof deterministic mechanisms. Such deterministic mechanisms are known to be dictatorships (cf. Gibbard, 1973; Satterthwaite, 1975). The question of whether such a decomposition is possible on restricted domains on which there are non-dictatorial unanimous and strategy-proof deterministic mechanisms has remained open. Using our tools, we answer it in the affirmative on the single-peaked voting domain. Deterministic strategy-proof and unanimous mechanisms on this domain were characterized by Moulin (1980) and have been studied intensively ever since. It turns out that strategy-proofness and unanimity, with and without anonymity, are decomposable even though they are not TUM.[7] This result is surprising given the observation by Ehlers, Peters,

[4]A deterministic mechanism is strategy proof if for every agent, submitting his true preference ranking is at least as good as submitting any other ranking irrespective of the preference rankings submitted by other agents. This is equivalent to ex-post incentive-compatibility. A random mechanism is strategy-proof if for every agent, submitting his true preference ranking first-order stochastically dominates submitting any other preference ranking irrespective of the preference rankings submitted by other agents. This is the standard notion of incentive-compatibility of ordinal random mechanisms introduced by Gibbard (1977, 1978); Roth and Rothblum (1999); Bogomolnaia and Moulin (2001).

[5]Unanimity is a weak form of efficiency. A mechanism is unanimous if, whenever there are outcomes that are among the most desirable choices for all agents then the mechanism implements one of these outcomes.

[6]A mechanism is anonymous if the outcome of the mechanism depends only on the set of preferences reported, not on who reported them.

[7]While proving this result, we also obtain a corollary to Moulin (1980) using our tops-onlyness results,

and Storcken (2002) that some strategy-proof and unanimous mechanisms cannot be decomposed on the same domain as a randomization over the particular subset of strategy-proof and unanimous deterministic mechanisms that they study. This paper's main contribution is the characterization of strategy-proof and unanimous random mechanisms in the single-peaked preference voting model. Unlike our approach, they come up with a random mechanism class that is not defined as a probability distribution over deterministic strategy-proof and unanimous mechanisms. Hence, our result also implies that the Ehlers, Peters, and Storcken (2002) class is equivalent to probability distributions over the Moulin (1980)'s unanimous subclass.

We introduce the decomposition tools in a unified model of many economic environments. In our model, there is a finite number of agents and a finite number of social and personalized outcomes. Agents have preferences over personal outcomes. The model encompasses voting, public goods provision, assignment of discrete goods with and without transfers, assignment of divisible goods, matching, coalition formation, and network formation. Each of these environments corresponds in the unified model to a set of conditions on what outcomes are feasible and a condition on the class of allowable ordinary preference profiles. The feasibility condition allows us to include both standard strict-preference voting problems (everybody obtains the same outcome) and object allocation (everybody obtains a different outcome). The preference domain condition allows us to include both environments without transfers (all preference profiles over outcomes are allowed) and environments with transfers.

In this unified model we primarily study ordinal mechanisms, that is, mechanisms whose message space consists of ordinal preference rankings over sure outcomes.[8] To draw on combinatorial integer programming, we represent the random mechanism as a vector of probabilities indexed by agents, agents' outcomes, and agents' preference profiles. We show that the feasibility of the mechanism (e.g., the sum of the probabilistic outcomes sum up to 1, or constraints implying this end) along with certain properties can be represented as a TUM matrix whose rows are indexed by agents, agents' outcomes, and agents' preference profiles, and columns correspond to these constraints. A matrix is TUM if all of its square submatrices have determinants equal to $-1$, 0, or 1. We represent the feasibility constraints by columns corresponding to each preference profile separately, over every pair of agent and agent's outcome. Certain properties, like individual rationality constraints, are also sepa-

the characterization of the full class of deterministic, unanimous, and strategy-proof voting mechanisms on a single-peaked domain (cf. Corollary 1). As far as we know, we are the first ones to give this full characterization.

[8] See Bogomolnaia and Moulin (2001) for a thorough discussion of the role of ordinal random mechanisms. Ordinal mechanisms in the presence of random outcomes were also studied by Gibbard (1978); Abdulkadiroğlu and Sönmez (1998).

rable across preference profiles. However, properties such as strategy-proofness are defined over preference profile pairs as well as agents and their outcomes. Provided the structure of feasibility constraints of the environment is also TUM, we show that TUM feasibility constraints and the individual-rationality constraint may be jointly represented by a TUM matrix. We also show that strategy-proofness and simple feasibility can be represented by a TUM constraint matrix in certain environments. Finally, we rely on the result of Hoffman and Kruskal (1956) to show that the real-valued vector that codes a random mechanism satisfying the constraint represented by the TUM matrix is equal to a probability-weighted sum of integer-valued vectors. Each of these integer-valued vectors represents a deterministic mechanism that satisfies the same feasibility constraints and properties.

A forerunner to our study is Sethuraman, Teo, and Vohra (2003, STV from now on). They used the theory of combinatorial integer programming to represent and decompose Arrovian social choice functions through linear constraints. The work of Budish, Che, Kojima, and Milgrom (2013, BCKM from now on) is also related to ours in that they use combinatorial integer programming to study decomposition problems related to matching. In contrast to our paper, both of these papers only study the question of whether we can decompose a particular random allocation into a randomization over deterministic outcomes while preserving constraints. This is an important question: the outcome of a random mechanism is a matrix of marginal probabilities that needs to be implemented through feasible deterministic outcomes. They restrict their attention to constraints expressible as an unweighted sum of probabilistic decision variables and allocation probabilities, respectively. Our setup allows richer, integer-weighted constraints on implementation of random allocations. Our constraint language is rich enough to study mechanism design, and, for instance, to express and study constraints such as strategy-proofness constraints that are not expressible in the language of STV and BCKM.[9]

Our paper is also related to Peters, Roy, Sen, and Storcken (2014), Chatterji, Roy, and Sen (2012), Picot and Sen (2012), and Chatterji, Sen, and Zeng (2014). Their and our papers are independent.[10] The closest to ours among these papers is Peters, Roy, Sen, and Storcken (2014). They show that on the single-peaked domain, every strategy-proof and unanimous random mechanism is a lottery over such deterministic mechanisms; their proof technique relies on Farkas Lemma and is different from our approach for the proof of Theorem 6. Chatterji, Roy, and Sen (2012) prove a decomposability property on lexicographic product

[9]BCKM show that their bi-hierarchy condition is not only sufficient but also necessary for decomposability; as demonstrated by our paper, their necessity result hinges on the restriction to the set of constraints they study, and in fact we can decompose constraints that fail the bi-hierarchy condition.

[10]We became aware of each other's work after both their and our papers were completed. We thank Arunava Sen for telling us about his and his co-authors' work.

domains; Picot and Sen (2012) prove it for the case of two social alternatives; and Chatterji, Sen, and Zeng (2014) show that the decomposability of strategy-proofness and unanimity does not hold in general.

## 2 Motivating Examples

We start by giving some examples of random mechanisms and the types of constraints and properties that are decomposable. Our results will generalize these examples.

We start with simple feasibility constraints of summation type over random outcomes, which our model covers.

**Example 1** Consider an environment in which a random assignment gives each agent a probability of assignment for each object such that each object and each agent cannot be given a total assignment probability greater than 1. This constraint is a typical *feasibility* constraint. It can be written as a linear summation saying that summations of all entries regarding any agent should not exceed 1 and all entries regarding any object should not exceed 1. In this environment, a social outcome is an assignment of objects to agents such that no two agents receive the same object, while a personal outcome is the object that an agent receives. It turns out that any such random assignment, which consists of marginal probability distributions over personalized outcomes subject to the feasibility constraint, is decomposable into a lottery over social outcomes. This is a famous result independently proven by Birkhoff (1946) and von Neumann (1953). The recent paper by BCKM generalizes this result and shows that as long as there are at most two (or bi-) *hierarchical summation* constraint sets regarding these entries, the random assignment can be decomposed into a lottery over deterministic assignments such that each also satisfies the same constraints. A hierarchical summation constraint set is a set of linear summation constraints over entries of the random assignment such that any two constraints in the set are either embedded in one another or are mutually exclusive from each other in regard to the entries they apply to. $\diamond$

By contrast, our representation is more general and handles not only bi-hierarchical summation constraints but also integer (or rational number) weighted summation constraints by constraints with $+1$ and $-1$ (i.e., summation and subtraction constraints). Thus, in the sequel, we focus on constraints with $+1$ and $-1$. Two constraints of this sort are individual rationality and strategy-proofness. An ordinal random mechanism allocates a probability distribution over a social outcome subject to the reported ordinal preferences of agents. An outcome of such a mechanism not only satisfies simple feasibility constraints (such as that the sum of outcome probabilities should be one), but usually also has certain properties that require linkages among different random outcomes such as its lottery outcomes regarding different preference profiles. In this vein, the first example demonstrates decomposability of the individual rationality property using TUM decomposition, which links an initial deterministic outcome to random outcomes obtained by the mechanism at different preference profiles.

The second property we explore is strategy-proofness. We give an example of a non-dictatorial decomposition of a strategy-proof mechanism on an individual choice subdomain of preferences, truncation choice models (see Subsection 4.3). It turns out that in this domain strategy-proofness is TUM. However, in other domains, this is not necessarily true, as we explore in Section 5.

# 3 Setup

## 3.1 Environments

Let $I$ be a finite set of agents. Let $O_i$ be a finite set of sure **personal outcomes** of agent $i \in I$. For instance, the sure outcome might be a social choice, voting result, an object the agent was assigned, or the assigned object and the price paid. Let $O \subseteq \times_{i\in I} O_i$ be the set of feasible profiles of sure **social outcomes**. We will use also the term **environment** to refer to $O$. We will derive our strongest positive results for **voting problems** where $O_1 = ... = O_{|I|}$ and $O = \left\{ \left(o_1, ..., o_{|I|}\right) \in \times_{i\in I} O_i \mid o_1 = ... = o_{|I|} \right\}$. Although by introducing indifferences we make this environment rich enough to embed house allocation, matching, and other problems we are interested in, the formulation of the results becomes more transparent when we build into the model the distinction between a personal outcome for agent $i$ and agent $j$. For instance, in the **house allocation problem**, we assume that $O_1 = ... = O_{|I|}$ and $O = \left\{ \left(o_1, ..., o_{|I|}\right) \in \times_{i\in I} O_i \mid \forall x_1 \in O_1 \ \left|\{i : o_i = x_1\}\right| \leq 1 \right\}$.

We consider randomizations over social outcomes as well as sure social outcomes. A **lottery** is a probability distribution over feasible social outcomes. Formally, a lottery $\mathcal{L}$=$(\mathcal{L}(o))_{o\in O}$ satisfies (1) for all $o \in O$, $\mathcal{L}(o) \in [0, 1]$, and (2) $\sum_{o\in O} \mathcal{L}(o) = 1$. By a slight abuse of notation, we will denote by $\mathcal{L}(i, o_i)$ the probability that agent $i$ obtains personal outcome $o_i \in O_i$ under lottery $\mathcal{L}$, i.e., $\mathcal{L}(i, o_i) = \sum_{x\in O\,:\,x_i=o_i} \mathcal{L}(x)$. By $\Delta O$ we will denote the set of lotteries over social outcomes.

Each agent $i \in I$ has preferences over personal outcomes drawn from a preference domain $D_i$. The domain is **universal** if every possible ranking of personal outcomes is possible. For any $\succsim_i \in D_i$ and $Y_i \subseteq O_i$, let $Ch(Y_i, \succsim_i) \subseteq Y_i$ denote the **choice set** of agent $i$ at preference $\succsim_i$ among the personal outcomes in $Y_i$, i.e., for all $o_i \in Ch(Y_i, \succsim_i)$, $o_i \succsim_i x_i$ for all $x_i \in Y_i$.

When the choice set is a singleton, we will occasionally treat it as the choice outcome by an abuse of notation. We refer to an environment $O$ and preference domain $D$ pair an **model**.

## 3.2 Mechanisms, Constraints, and Decomposability

Fix a model $(O, D)$. A(n **ordinal**) **random mechanism** $\phi : D \to \Delta O$ is a mapping from preference profiles to lotteries over feasible profiles of sure social outcomes. By $\phi(\succsim; o)$ we will denote the probability that social outcome $o$ is chosen by $\phi$ at preference profile $\succsim$. A mechanism is deterministic if it maps preference profiles into $O$. That is, a deterministic mechanism $\phi$ is such that for all $\succsim \in D$, $\phi(\succsim, o) = 1$ for some $o \in O$. We will sometimes denote this by slight abuse of notation as $\phi(\succsim) = o$. For a general random mechanism, by $\phi(\succsim; i, o_i)$ we will denote the probability that personal outcome $o_i \in O_i$ is being chosen for agent $i$ at $\succsim$; that is, $\phi(\succsim; i, o_i) = \sum_{x \in O \,:\, x_i = o_i} \phi(\succsim; x)$. Let $\mathcal{IO} = \{(i, o_i) \mid i \in I \text{ and } o_i \in O_i\}$ be the set of feasible agent and individual outcome pairs.

We study mechanisms $\phi$ that **satisfy** a number of *elementary constraints* of the following form

$$\underline{c} \leq \sum_{(\succsim, i, o_i) \in C} \phi(\succsim; i, o_i) - \sum_{\left(\succsim', i', o'_{i'}\right) \in C'} \phi(\succsim'; i', o'_{i'}) \leq \bar{c},$$

where $C, C' \subset D \times \mathcal{IO}$ (one of them might be empty) and $\underline{c}, \bar{c} \in \mathbb{Z} \cup \{-\infty, +\infty\}$. An **elementary constraint** is thus identified with a quadruple $(C, C', \underline{c}, \bar{c})$. We define a *constraint* as a conjunction of elementary constraints. A **constraint** $\mathcal{C}$ is thus identified as a finite set of elementary constraints as $\mathcal{C} = \{(C_j, C'_j, \underline{c}_j, \bar{c}_j)\}_{j=1}^m$. A mechanism **satisfies** a constraint $\mathcal{C}$ if it satisfies each elementary constraint in $\mathcal{C}$.

We say that a random mechanism $\phi$ is **decomposable under** a constraint $\mathcal{C}$ if $\phi$ satisfies $\mathcal{C}$ and there exists a finite set of deterministic mechanisms $\{\psi_i\}_{i=1}^k$ and non-negative weights $\{\alpha_i\}_{i=1}^k$ such that (i) $\sum_{i=1}^k \alpha_i = 1$ and $\phi(\succsim, o) = \sum_{i=1}^k \alpha_i \psi_i(\succsim, o)$ for all $\succsim \in D$ and $o \in O$ and (ii) each $\psi_i$ satisfies $\mathcal{C}$ for all $i \in \{1, ..., k\}$. A constraint $\mathcal{C} = (C, C', \underline{c}, \bar{c})$ is **decomposable** if every random mechanism $\phi$ is decomposable under $\mathcal{C}$**.**

The above class of constraints is surprisingly versatile. For instance, strategy-proofness belongs to this class of constraints. An ordinal mechanism $\phi$ is **strategy-proof** if for every agent $i$ and every profile of preferences reported by other agents, the distribution of agent $i$'s outcomes when he reports his preferences truthfully first-order stochastically dominates the distribution of outcomes from reporting any other preference profile.[11] We write the *strategy-proofness constraint* in our language of elementary constraints as follows:

[11] See Bogomolnaia and Moulin (2001). One interpretation of the condition is that there is no profile of other agents' reports, and no cardinal representation of $\succsim_i \in D_i$ that allows an agent $i$ to submit a non-truthful $\succsim_i$ and improve his expected payoff from submitting the true preferences.

for all $i \in I$, $\succsim_{-i} \in \times_{j \neq i} D_j$, $\succsim_i, \succsim'_i \in D_i$ and $x_i \in O_i$,

$$0 \leq \sum_{o_i \succsim_i x_i} \phi(\succsim_i, \succsim_{-i}; i, o_i) - \sum_{o_i \succsim_i x_i} \phi(\succsim'_i, \succsim_{-i}; i, o_i) \leq 1. \tag{1}$$

The two above examples of feasibility constraints are also expressible in terms of elementary constraints. Thus, a mechanism $\phi$ satisfies the *feasibility constraint of the house allocation problem* if it satisfies

$$0 \leq \phi(\succsim; i, x_1) \leq 1 \quad \forall i \in I,\ \forall x_1 \in O_1,\ \forall \succsim \in D, \tag{2}$$

$$\sum_{x_1 \in O_1} \phi(\succsim; i, x_1) = 1 \quad \forall i \in I,\ \forall \succsim \in D, \tag{3}$$

$$0 \leq \sum_{i \in I} \phi(\succsim; i, x_1) \leq 1 \quad \forall x_1 \in O_1,\ \forall \succsim \in D.$$

A mechanism $\phi$ satisfies the *feasibility constraint of the voting problem* if it satisfies Equations 2 and 3 above, and

$$\phi(\succsim; i, x_1) - \phi(\succsim; j, x_1) = 0 \quad \forall i, j \in I,\ \forall x_1 \in O_1,\ \forall \succsim \in D.$$

# 4 Totally Unimodular Decomposition

We define the **incidence matrix** of a constraint $\mathcal{C}$ as a matrix whose rows are indexed by $(\succsim, i, o)$ and columns indexed by the first two coordinates $(C, C')$ of elementary constraints in $\mathcal{C}$. Without loss of generality, we assume that constraint $\mathcal{C}$ contains at most one elementary constraint with a given pair of the first two coordinates $(C, C')$. The value of a cell is 0 if it does not belong to $C \cup C'$, $+1$ if it belongs to $C$, and $-1$ if it belongs to $C'$. We thus impose asymmetric roles on sets $C$ and $C'$ even though they may play symmetric roles. The way in which we order them turns out to be insubstantial: multiplying any column by $-1$ will not affect any of the statements below.

Our approach to feasibility constraints relies on earlier results on *totally unimodular* integer matrices. A matrix is **totally unimodular** (**TUM**) if the determinant of each of its square submatrices is either $-1$, 0, or 1. We will rely on a useful characterization of TUM provided by Ghouila-Houri (1962):

**Lemma 1 (Ghouila-Houri (1962) TUM Matrix Decomposition Lemma)** *A matrix $A \in \{-1, 0, 1\}^{m \times n}$ is totally unimodular if and only if every subset $S$ of rows may be partitioned into sets $S^+$ (we will also refer to this as the set of "blue" rows) and $S^-$ (we will also* *refer to this as the set of "red" rows) such that for every column $c$ of $A$:*[12]

$$\sum_{r\in S^+} A(r,c) - \sum_{r\in S^-} A(r,c) \in \{-1,0,+1\}. \tag{4}$$

We refer to $\{S^+, S^-\}$ as the **Ghouila-Houri partition of** $S$ and the difference in Equation 4 as the **Ghouila-Houri difference along** $c$.[13] We call a constraint **totally unimodular** if it is has a TUM incidence matrix.

Our first theorem is related to the sufficiency of TUM decomposability in decomposing properties.

**Theorem 1** *If a random mechanism $\phi : D \to \Delta O$ satisfies a TUM constraint, then it is a convex combination of a finite number of deterministic mechanisms $\{\phi_k : D \to O\}$ each of which satisfies the same constraint, i.e., there is a finite number of weights $\{\lambda_k\} \subset [0,1]$ with $\sum_k \lambda_k = 1$ such that $\phi = \sum_k \lambda_k \phi_k$. If a mechanism $\phi : D \to \times_{i\in I}\mathbb{R}^{|O_i|}$ satisfies a TUM constraint, then it is a convex combination of finitely many integer-valued mechanisms $\{\phi_k : P \to \times_{i\in I}\mathbb{Z}^{|O_i|}\}$ each of which satisfies the constraint.*

**Proof of Theorem 1.** Because the constraint is TUM, it has a TUM incidence matrix. Let us call that matrix $A$. Consider a row vector $\phi$ of real numbers, one for each triple: agent $i$, outcome $o_i$ of agent $i$, and preference profile of all agents $\succsim$ such that the cell $(\succsim, i, o_i)$ of the vector equals $\phi(\succsim; i, o_i)$. Since $\phi$ satisfies the constraint, $\phi A$ is a row vector indexed by the first two coordinates of the constraint $(C, C')$, and for each constraint $(C, C', \underline{c}, \bar{c})$ the relevant entry in $\phi A$ is between the relative $\underline{c}$ and $\bar{c}$. By Hoffman and Kruskal (1956), the set of all vectors with this property is an integral polyhedron: that is, it is a polyhedron with integer-valued extreme points. Since it is convex, we may represent $vA$ as a convex combination of the extreme points. Each of the extreme points represents a $\{0,1\}$ −valued mechanism $\phi_k$. Since the extreme point belongs to the polyhedron, the mechanism satisfies the constraints represented by $A$. The proof of the second statement follows the same lines. ■

Below we give three examples of TUM properties.

## 4.1 Feasibility

Observe that a number of feasibility constraints are TUM. Ghouila-Houri decomposition immediately implies that voting and house allocation feasibility constraints are TUM. BCKM

[12] We can also state this result by interchanging the roles of rows with columns.
[13] We give a detailed example of the construction of such a partition in Example 3 below.

gave examples of feasibility constraints that are defined over a random outcome on the house allocation domain. Ghouila-Houri decomposition can be used to prove their results. We can extend this result to feasibility constraints for mechanisms that not only pertain to a random outcome but also link random outcomes generated by different preference profiles. As feasibility is the fundamental property for defining a mechanism and its decomposition, we will implicitly assume from now on that all mechanisms and their decompositions have to satisfy this property, possibly in addition to other properties.

## 4.2 Individual Rationality

Theorem 1 allows us to decompose random mechanisms while preserving individual rationality. To define individual rationality we need to enrich the model by identifying a status quo outcome $\omega_i \in O_i$, an outside option for each agent $i \in I$ such that $\omega = (\omega_1, ..., \omega_{|I|}) \in O$. A mechanism $\phi$ is **individually rational** if each agent's outcome first-order stochastically dominates the status quo outcome; that is, for all $i \in I$ and $\succsim \in D$, $\omega_i \succ_i o_i \Rightarrow \phi(\succsim; i, o_i) = 0$. Our main result for this subsection is as follows:

**Theorem 2** *The conjunction of individual rationality and any TUM feasibility constraint is TUM in any model. In particular, individual rationality is decomposable in any model.*

**Proof of Theorem 2.** Individual rationality is a conjunction of simple constraints denoted by $\mathcal{IR} = \{(C, C') = (\{(\succsim, i, o_i)\}, \emptyset) \mid \omega_i \succ_i o_i\}$. Lemma 1 and Theorem 1 imply that any set of rows of feasibility may be partitioned into Ghouila-Houri sets $S^+$ and $S^-$. Since each individual rationality column contains exactly one non-zero element, the same Ghouila-Houri partition establishes that feasibility together with individual rationality is TUM. ■

Below, we illustrate this result with an example:

**Example 2** Consider an environment in which there are two agents with preferences over a number of discrete objects such as dormitory rooms, and each of them is initially endowed with a distinct room. Some rooms are not initially endowed to anybody. Consider the following direct mechanism that probabilistically distributes rooms based on the agents' reported preferences:

- each agent will receive his first-choice room with probability 1, if they are distinct;
- if both agents rank the same room as their first choice, which is endowed to one of the agents, then the owner of that room will receive it with probability 1 and the other agent will receive his second-choice room with certainty; and

- if both agents rank the same room as their first choice and this room is not initially endowed to anybody, then an even lottery will determine who receives it, and the second choice of each agent will be assigned to him with probability $\frac{1}{2}$.

This mechanism is individually rational, as nobody will receive a room strictly worse than his endowment with a positive probability.

Using our methodology, the incidence matrix $A$ of individual rationality and simple feasibility constraints is as follows: rows of $A$ refer to each (preference profile, agent, room) triple $(\succsim, i, o_i)$; columns are for feasibility and individual rationality constraints: the first part of the feasibility constraint columns is indexed for each (preference profile, room) pair $(\succsim, o_i)$ (We refer to them as type (a) columns) and for each preference profile, agent pair $(\succsim, i)$ (type (b) columns); and the individual rationality constraints and the remaining feasibility constraints are indexed for each (preference profile, agent, room) triple $(\succsim, i, o_i)$ (type (c) columns). Let $(\omega_i)_{i\in I} \in O$ be the endowment. Let row vector $\phi$ refer to the above mechanism where each entry refers to the probability of assignment for each (preference profile, agent, room) triple. Then all constraints are denoted by $\underline{c} \leq \phi A \leq \overline{c}$ for row vectors $\underline{c}$ and $\overline{c}$ where for all $(\succsim, i, o_i)$, we have

$$A[(\succsim, i, o_i), (\succsim', o_i')] = \begin{cases} 1 & \text{if } o_i = o_i' \text{ and } \succsim = \succsim' \\ 0 & \text{otherwise} \end{cases} \quad \forall (\succsim', o_i'),$$

$$A[(\succsim, i, o_i), (\succsim', i')] = \begin{cases} 1 & \text{if } i = i' \text{ and } \succsim = \succsim' \\ 0 & \text{otherwise} \end{cases} \quad \forall (\succsim', i'),$$

$$A[(\succsim, i, o_i), (\succsim', i', o_i')] = \begin{cases} 1 & \text{if } i = i', o_i = o_i', \text{ and } \succsim = \succsim' \\ 0 & \text{otherwise} \end{cases} \quad \forall (\succsim', i', o_i'); \text{ and}$$

$$\underline{c}[(\succsim, o_i)] = 1 \quad \text{and} \quad \overline{c}[(\succsim, o_i)] = 1 \quad \forall (\succsim, o_i),$$

$$\underline{c}[(\succsim, i)] = 1 \quad \text{and} \quad \overline{c}[(\succsim, i)] = 1 \quad \forall (\succsim, i),$$

$$\underline{c}[(\succsim, i, o_i)] = 0 \quad \text{and} \quad \overline{c}[(\succsim, i, o_i)] = \begin{cases} 1 & \text{if } o_i \succsim_i \omega_i \\ 0 & \text{otherwise} \end{cases} \quad \forall (\succsim, i, o_i).$$

It is straightforward to verify that $A$ is TUM. Using the Ghouila-Houri decomposition method for columns instead of rows, for any subset of columns $S$, we can color them blue (i.e., assign to subset $S^+$) or red (i.e., assign to subset $S^-$) as follows: Color all type (a) constraints in $S$ blue and all type (b) constraints red. Consider type (c) constraints in $S$. Observe that type (c) constraints apply only to a single row for each column (i.e., positive only at one cell), i.e., the cell indexed by the same row and column dimensions. One can color the type (c) columns of $S$ as follows: Check the difference in sums of blue and red columns of type (a) and type (b) along a row $(\succsim, i, o_i)$ that is also a type (c) column index in $S$. This is either $-1, 0,$ or 1, as there is at most one blue and one red column already colored with a positive entry along each row in $S$. If the difference is $-1$, color column $(\succsim, i, o_i)$ blue, if the difference is 1 then color it red, and if the difference is 0 color it any color, for example blue. Now the difference of blue and red columns in $S$ with positive entries along a row is 0 or 1, and we are done. By Theorem 1, $\phi$ is decomposable into individually rational deterministic mechanisms.

It turns out that we can decompose $\phi$ as follows: Consider two priority orders of agents, one ranking agent 1 over 2 and the other ranking agent 2 over 1. Based on one of these priority orderings, which is selected with an even lottery, each agent is assigned, in order, his top-choice available house, provided that the top choice of the first agent in the order is not the endowment of the second agent who also ranks it first. In this case, the second agent is assigned his first choice and the first agent is assigned his second choice.[14] $\diamond$

## 4.3 Strategy-proofness

Strategy-proofness constraints also turn out to be TUM in certain cases. We provide an example of an individual choice model where strategy-proofness is TUM.[15] We consider an agent whose rankings of outcomes are known and common knowledge. However, he can potentially *truncate* his preferences by declaring some choices unacceptable. This model is consistent with Simon's (1955; 1956) model of *satisficing*, which is a decision-making strategy that involves searching through the available alternatives until a satisfaction threshold is met. We formally introduce a truncation decision domain as follows. Let $I = \{i\}$, and $O^*$ be an ordered finite set of sure real outcomes of agent $i$; denote the ordering by $\sqsupset$. Without loss of

[14] These deterministic mechanisms are in the class of *top-trading cycles* mechanisms (cf. Abdulkadiroğlu and Sönmez, 1999; Pápai, 2000).

[15] For strategy-proofness in collective choice situations, see Section 5.

generality let $O^* = \{o_1, ..., o_n\}$ and $o_1 \sqsupset o_2 \sqsupset ... \sqsupset o_n$. This refers to the common-knowledge strict ranking of the agent regarding the real outcomes. For instance, the sure real outcome might be a social choice, voting result, an object the agent was assigned, or the assigned object and the price paid. The set of all sure outcomes is given by $O = O^* \cup \{o_0\}$, where $o_0$ is the satisfaction bound. It can be an outside option in models with individual rationality. A truncation preference relation $\succsim$ is a strict ranking over $O$ such that $o_k \succ o_{k+1}$ for all $k \in \{1, 2, ..., n-1\}$. Hence, two difference truncation preferences differ from each other only where $o_0$ is ranked. Each agent $i$ is endowed with a strict ranking $<_i$ over $O_i$, and a satisfactory threshold $o_i$. Given an outcome $o_k$ for $k \in \{0, ..., n\}$, the preference relation $\succsim^k$ is the one with $o_0$ ranked just below $o_k$ if $k \geq 1$ and $o_0$ ranked at the top if $k = 0$.

We consider mechanisms that elicit the agent's threshold and map it into lotteries over sure outcomes. In this environment, we will see that — with a single agent — if the feasibility constraints are totally unimodular, then the incentive compatibility and feasibility constraints are totally unimodular. Hence, our main results imply that each incentive-compatible random mechanism can be decomposed as a lottery over incentive-compatible deterministic mechanisms.

We also care about *individual rationality*; mechanisms that select only random outcomes that are at least as good as $o_0$ .

We consider *simple feasibility* constraints: the sum of the probabilities over all outcomes is equal to one for an agent and each probability is non-negative.

**Theorem 3** *For a truncation decision model, the strategy-proofness constraint with or without an individual rationality constraint is TUM together with the simple feasibility constraint. In particular, strategy-proofness is decomposable in this model.*

Before the proof, let us mention that although TUM is a sufficient condition for decomposability of a constraint, it is not necessary. For instance, strategy-proofness and unanimity along with feasibility are non–TUM constraints even on subdomains of the universal domain such as single-peaked preferences (hence they are not TUM in the universal domains, either).

**Proof of Theorem 3.** Rewriting Equation 1 for a single agent $I = \{i\}$ by dropping the $i$ subscript, strategy-proofness is a conjunction of simple constraints of the form $\succsim, \succsim' \in D$, where $D$ is the truncation preference domain and $x \in O$,

$$0 \leq \sum_{y \succsim x} \phi(\succsim, y) - \sum_{y \succsim x} \phi(\succsim'; y) \leq 1.$$

This is equivalent to, for all $k \in \{0, 1, ..., n-1\}$,

$$\begin{aligned}
0 &= \phi(\succsim^k; o_\ell) - \phi(\succsim^{k+1}; o_\ell), \quad \forall \ell \in \{1, 2, ..., k\}; \\
0 &\leq \phi(\succsim^{o_k}; o_0) - \phi(\succsim^{o_{k+1}}; o_0) \leq 1; \\
-1 &\leq \phi(\succsim^{o_k}; o_{k+1}) - \phi(\succsim^{o_{k+1}}; o_{k+1}) \leq 0; \\
0 &= \phi(\succsim^{o_k}; o_\ell) - \phi(\succsim^{o_{k+1}}; o_\ell), \quad \forall \ell \in \{k+2, ..., n\}.
\end{aligned} \tag{5}$$

That is, we can inspect only local deviations in the satisfaction threshold, if we would like to capture all deviations. Thus, we incorporate strategy-proofness constraints in Equation 5 in the constraint matrix $A$: Consider $C = (\succsim^k, x)$ and $C' = (\succsim^{k+1}, x)$ for all $k \in \{0, ..., n-1\}$ and $x \in O$: for column $c = (C, C')$, at row $r = (\succsim^k, x)$, $A(r, c) = 1$ and at row $r' = (\succsim^{o_{k+1}}, x)$, $A(r', c) = -1$, respectively.

On the other hand, simple feasibility and individual rationality constraints are of the form: for all $k \in \{0, 1, ..., n\}$,

$$\begin{aligned}
&0 \leq \phi(\succsim^k; o_\ell) \leq 1 \quad \forall \ell \in \{0, 1, ..., k\}, \\
&0 = \phi(\succsim^k; o_\ell) \quad \forall \ell \in \{k+1, k+2, ..., n\}, \text{ and} \\
&1 = \sum_{x \in O} \phi(\succsim^k; x).
\end{aligned} \tag{6}$$

If there were no individual rationality constraints, then the second line of Equations 6 would be an inequality between 0 and 1. Thus, with or without individual rationality, the same $A$ constraint matrix is formed: for all columns of the form $c = (\succsim^k; x)$ and rows of the form $r = c$, $A(r, c) = 1$; for all the columns of the form $c = (\succsim^k)$ and rows of the form $r = (\succsim^k, x)$, we have $A(r, c) = 1$.

We are ready to introduce a Ghouila-Houri (1962) decomposition for $A$: Take any subset $S$ of rows of $A$. In constructing a Ghouila-Houri partition of $S$, $\{S^+, S^-\}$, our goal is to pair rows $(\succsim, x), (\succsim, y) \in S$ for each $\succsim \in D$ such that

- **C1.** Rows $(\succsim, x) \in S$ and $(\succsim, y) \in S$ will be placed in opposite sets, $S^+$ and $S^-$, or $S^-$ and $S^+$, respectively.
- **C2.** All rows in $\{(\succsim', x) \in S \mid \succsim' \in D\}$ will be placed in the same set, $S^+$ or $S^-$.

We may not be able to perfectly pair all rows for a profile $\succsim$, as there can be an odd number of rows in $\{(\succsim, o) \in S \mid o \in O\}$; for such an outcome we impose only C2 above:

- **C2.** If there is an unpaired row $(\succsim, o) \in S$, then all rows in $\{(\succsim', o) \in S \mid \succsim' \in D\}$ will be placed in the same set in the partition, $S^+$ or $S^-$.

Our goal is the achieve the following for the all three types of columns of the constraint matrix $A$ if rows $(\succsim, x), (\succsim, y) \in S$ are paired together or $(\succsim, x) \in S$ is left unpaired.

**Type 1.** As we proceed down a column $(\succsim, x)$: the Ghouila-Houri difference will be exactly 1 or $-1$, for the whole column, since there is only one cell with a non-negative entry, i.e., the one corresponding to row $(\succsim, x)$.

**Type 2.** As we proceed down a column $(\succsim)$: if the two rows $(\succsim, x)$ and $(\succsim, y)$ are paired together, the Ghouila-Houri difference will cancel out to zero for these two rows, since these belong to opposite sets, and they each have cell value 1. On the other hand, if row $(\succsim, x) \in S$ is left unpaired, the Ghouila-Houri difference will be $-1$ or 1 for the row $(\succsim, x)$. (Achieved by C1.)

**Type 3.** As we proceed down a column $((\succsim^k, x), (\succsim^{k+1}, x))$ such that $\{\succsim, \succsim'\} = \{\succsim^k, \succsim^{k+1}\}$: if $(\succsim', x) \in S$, then the Ghouila-Houri difference will cancel out to zero for the whole column, since these belong to the same set, $S^+$ or $S^-$, and they have cell values $+1$ and $-1$, respectively. On the other hand if $(\succsim', x) \notin S$, then the Ghouila-Houri difference will be $-1$ or $+1$ for the whole column. (Achieved by C2.)

Observe that pairing will be crucial for Type 2 and Type 3 columns above. Moreover, because of Type 2 of columns $(\succsim)$, we need the pairing to be mutually exclusive: i.e., if $(\succsim, x)$ and $(\succsim, y)$ are paired together, then there is no $z \neq y$ such that also $(\succsim, x)$ and $(\succsim, z)$ are paired together; otherwise we cannot achieve our goal of the Ghouila-Houri difference being $-1, 0,$ or 1 along the whole column $(\succsim)$.

We construct an undirected graph $\mathcal{G} = (S, E)$ with nodes $S$ and edges $E$: Each node consists of a row in $S$. For all $k$ and $x \in O$, if $(\succsim^k, x)$,$(\succsim^{k+1}, x) \in S$, then we place an edge between nodes $(\succsim^k, x)$ and $(\succsim^{k+1}, x)$. This edge signifies that these rows should go to the same set of the partition, $S^+$ or $S^-$, to be determined later.

We *pair* the rows in $S$, i.e., nodes of $\mathcal{G}$, as follows, iteratively. Initially, we set counter $k' = 0$ and set $\mathcal{G}^0 = \mathcal{G}$.

**Step $k'$ for pairing of nodes of $\mathcal{G}$:**

Order the nodes of $\mathcal{G}^{k'}$ in $S^{k'} = \{(\succsim^{k'}, x) \in S \mid x \in O\}$ according to the length of the paths in $\mathcal{G}^{k'}$ that they initiate. Let $(\succsim^{k'}, x^1), (\succsim^{k'}, x^2), ..., (\succsim^{k'}, x^\ell)$ be this ordering such that $(\succsim^{k'}, x^1)$ initiates the longest path,$(\succsim^{k'}, x^2)$ initiates the second longest path, and so on (if there is a tie, it is broken arbitrarily). Pair nodes in these encountered paths as follows: for all odd $p$ $(\succsim^{k'}, x^p)$&$(\succsim^{k'}, x^{p+1})$,...,$(\succsim^k, x^p)$&$(\succsim^k, x^{p+1})$ are paired (where & shows the pairings) such that $k$ is the largest possible index satisfying the following two conditions:

- $(\succsim^k, x^p)$ is on the path initiated by $(\succsim^{k'}, x^p)$, and

- $(\succsim^k, x^{p+1})$ is on the path initiated by $(\succsim^{k'}, x^{p+1})$.

If $\ell$ (the number of the nodes in $S^{k'}$) is odd, an unpaired node $(\succsim^{k'}, x^\ell) \in S$ will remain involving preference profile $\succsim^{k'}$.

Remove all paired nodes, node $(\succsim^{k'}, x^\ell)$ if it was left unpaired, and all of their edges from $\mathcal{G}^{k'}$ to form the new graph $\mathcal{G}^{k'+1}$. Continue with the new graph as above by setting $k' := k' + 1$.

We construct another auxiliary graph $\Gamma = (S^*, E^*)$ from $\mathcal{G} = (S, E)$ as follows using the above pairings.

Whenever two nodes $r_1, r_2 \in S$ are paired in $\mathcal{G}$, construct a single node from them named $r_1\&r_2 \in S^*$. Any unpaired node $r \in S$ becomes a node $r \in S^*$. Take an edge $(r_1, r_2) \in E$ between nodes $r_1$ and $r_2 \in S$ and place an edge between the node including $r_1$ and the node including $r_2$ in $E^*$ as follows:

- If $r_1\&r \in S^*$ for some $r$ and $r_2\&r' \in S^*$ for some $r$ then place an edge between these two pairs in $\Gamma$, i.e., $(r_1\&r,\ r_2\&r') \in E^*$.

- If $r_1 \in S^*$ and $r_2\&r \in S^*$ for some $r$ then place an edge between them in $\Gamma$, i.e., $(r_1,\ r_2\&r) \in E^*$.

- If $r_1\&r \in S^*$ for some $r$ and $r_2 \in S^*$ then we place an edge between them in $\Gamma$, i.e., $(r_1\&r,\ r_3) \in E^*$.

We are ready to finish the proof of the theorem through two claims:
***Claim 1.*** $\Gamma$ has no cycles.
***Proof.*** Suppose that we form $\Gamma$ step by step as we pair the nodes in $\mathcal{G}$. We start with $\Gamma = \mathcal{G}$ and update $\Gamma$ as we pair new nodes at every step of the pairing process.

Suppose there was no cycle in $\Gamma$ until the pairing $(\succsim^k, x)\&(\succsim^k, y)$ was done in some Step $k' \leq k$, but to the contrary of the claim, this pairing has resulted with a cycle in $\Gamma$ as we updated it. For each $\{a, b\} = \{x, y\}$, let $P^a$ be the tree of nodes (i.e., the component without a cycle) in $\Gamma$ that $(\succsim^k, a)$ is located prior to being paired with $(\succsim^k, b)$. Now pairing of $(\succsim^k, x)$ with $(\succsim^k, y)$ causes a cycle in $\Gamma$ that traverses *backward* through the preference profiles $\succsim^{k-1}, ..., \succsim^{k-k''+1}, \succsim^{k-k''}$ and then traverses *forward* $\succsim^{k-k''+1}, ..., \succsim^{k-1}, \succsim^k$ for some $k''$. Suppose $(\succsim^{k-k''}, a)\&(\succsim^{k-k''}, b)$ is the node of $\Gamma$ in this cycle for the preference profile $\succsim^{k-k''}$. Suppose $(\succsim^{k-k''+1}, a)\&(\succsim^{k-k''+1}, c)$ and $(\succsim^{k-k''+1}, b)\&(\succsim^{k-k''+1}, d)$ are the two distinct nodes of $\Gamma$ in the same cycle (observe that one should have $a$ and the other one should have $b$ to be connected to the node $(\succsim^{k-k''}, a)\&(\succsim^{k-k''}, b))$. But then, this contradicts the pair construction: since $(\succsim^{k-k''}, a)\&(\succsim^{k-k''}, b)$ are already paired and we have both $(\succsim^{k-k''+1}, a), (\succsim^{k-k''+1}, b) \in S$, we should have also paired $(\succsim^{k-k''+1}, a)\&(\succsim^{k-k''+1}, b)$. $\diamond$

***Claim 2.*** Take a component of $\Gamma$. The two rows in each of its pair nodes can be assigned to $S^+$ and $S^-$, respectively, and its individual row nodes can be arbitrarily assigned to $S^+$ or $S^-$ such that C1 and C2 hold.

***Proof.*** Each component of $\Gamma$ can be represented as a tree, since there is no cycle. Suppose that the node with the largest indexed preference profile is taken as the root of the tree. Moreover, each paired node in the tree $(\succsim^k, a)\&(\succsim^k, b)$ has at most three edges connected to it in $\Gamma$: one of which can be an unpaired row node $(\succsim^{k+1}, a)$ or a paired rows node $(\succsim^{k+1}, a)\&(\succsim^{k+1}, x)$ and at most two nodes: (1) $(\succsim^{k-1}, a)\&(\succsim^{k-1}, y)$ or $(\succsim^{k-1}, a)$, and (2) $(\succsim^{k-1}, b)\&(\succsim^{k-1}, z)$ or $(\succsim^{k-1}, b)$.

We use induction in the construction of $S^+$ and $S^-$:

- Start with the root of the tree: If it is a paired rows node $(\succsim^{k'}, a)\&(\succsim^{k'}, b)$, then assign $(\succsim^{k'}, a)$ to $S^+$ and $(\succsim^{k'}, b)$ to $S^-$. If it is an unpaired row node $(\succsim^{k'}, a)$, then assign it to $S^+$. Observe that both C1 and C2 hold for this assignment.

- Assuming that we have already assigned all the nodes in the tree with preference profiles $\succsim^{k'}, \succsim^{k'-1}, ..., \succsim^{k+1}$ consistent with C1 and C2, we make the assignment of a node with preference profile $\succsim^k$ as follows:

  - If it consists of paired rows $(\succsim^k, x)\&(\succsim^k, y)$, then for $a \in \{x, y\}$, $(\succsim^{k+1}, a)$ is in the tree as well either by itself or in a pair. Suppose $(\succsim^{k+1}, a)$ was assigned to $S^\circ$ for some $\circ \in \{+, -\}$. Then we assign $(\succsim^k, a)$ to $S^\circ$ (so C2 holds for these two rows) and assign $(\succsim^k, b)$ to $S^\bullet$ for $b \in \{x, y\} \setminus \{a\}$ and $\bullet \in \{+, -\} \setminus \{\circ\}$ (so C1 holds for these two rows). If $(\succsim^{k+1}, b)$ is also in the tree, then it was paired with $(\succsim^{k+1}, a)$ by construction. Therefore, $(\succsim^{k+1}, b) \in S^\bullet$ by the inductive assumption, as $(\succsim^{k+1}, a) \in S^\circ$. Hence, this is consistent with $(\succsim^k, b) \in S^\bullet$ (so C2 holds for these two rows).
  - If it consists of unpaired row $(\succsim^k, x)$, then $(\succsim^{k+1}, x)$ is in the tree as well. And if $(\succsim^{k+1}, x) \in S^\circ$ for some $\circ \in \{+, -\}$, we assign $(\succsim^k, x) \in S^\circ$.

This completes the proof of the claim.$\diamond$

As C1 and C2 are simultaneously satisfied for all components of $\Gamma$, $A$ is TUM by Lemma 1 and hence the constraints are decomposable by Theorem 1. ■

The example below illustrates the Ghoulia-Houri partition construction used in the proof:

**Example 3** Consider an individual choice truncation problem with nine real outcomes $\{o_1, ...o_9\}$ and an outside option $o_0$. There are ten preference profiles as $\succsim^k$, where $o_k$ being the last acceptable real choice for $k \geq 1$ and $\succsim^0$ has no acceptable real outcome. Consider a set of rows of the constraint matrix $A$, denoted by $S$. The graph $\mathcal{G}$ as defined in the proof of Theorem 3 is given in Figure 3.

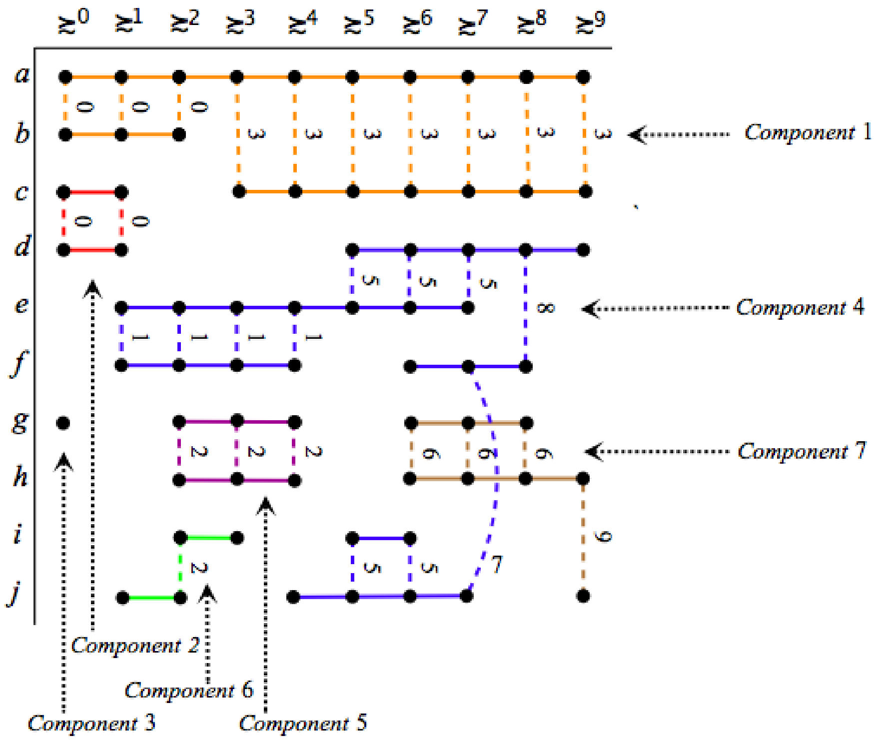


Figure 1: Nodes are the rows in $S$. Graph $\mathcal{G}$ is denoted by solid edges. Pairings are denoted by dashed edges.

Nodes of $\mathcal{G}$ consist of the rows in $S$, and edges of $\mathcal{G}$ are drawn with solid lines. For labeling the nodes, the vertical axis is indexed by all ten outcomes reindexed as $a, b, ..., j$, and the horizontal axis is indexed by the preferences profiles in order (e.g., node in cell $(\succsim^3, c)$ means $(\succsim^3, c) \in S$ and no node in cell $(\succsim^2, c)$ means $(\succsim^2, c) \notin S$ ). If we have two consecutive nodes horizontally in the figure for an outcome $x$ along preference profiles $\succsim^k$ and $\succsim^{k+1}$, (e.g., at $(\succsim^1, f)$ and $(\succsim^2, f)$; however, observe that $(\succsim^2, j)$ and $(\succsim^4, j)$ do not fulfill this requirement, as node $(\succsim^3, j) \notin S$), then these two rows have a 1 and a $-1$ respectively in the column $((\succsim^k, x), (\succsim^{k+1}, x))$ of $A$. For the Ghoulia-Houri difference for these two rows to neutralize to 0, we need both of these rows to be in the same set $S^+$ or $S^-$ (to be determined later).

Hence, the horizontal solid edges between two consecutive nodes in the figure refer to this requirement, which is condition C2 in the proof.

We also add some dashed vertical edges – denoting the pairing of rows in the proof – to satisfy condition C1. These can be added in a number of ways. The proof illustrates only one way of adding those. More specifically, dashed vertical edges are drawn between two nodes with the same preference profile. This means that, as these rows have entry 1 along the column $(\succsim^k)$ of $A$, we need them to belong to the two opposite sets $S^+$ and $S^-$, respectively. Hence, the Ghoulia-Houri difference neutralizes to 0 for these two rows along column $(\succsim^k)$ of $A$. (Which one will go to set $S^+$ and which one will go to set $S^-$ will be determined later.) This requirement is denoted as C1 in the proof.

Now, can we achieve these two goals C1 and C2 simultaneously for all rows in S that are linked through solid edges? The dashed edges will be added specifically to make this possible using the pairing procedure in the proof.

In Step 0, we pair nodes vertically in the two longest solid-edge paths initiated by $\succsim^0$: $(\succsim^0, a)\&(\succsim^0, b)$, $(\succsim^1, a)\&(\succsim^1, b)$, $(\succsim^2, a)\&(\succsim^2, b)$ as in the figure. We continue with the third- and fourth-longest solid-edge paths: $(\succsim^0, c)\&(\succsim^0, d)$, $(\succsim^1, c)\&(\succsim^1, d)$. The fifth row with $\succsim^0$ in $S$, $(\succsim^0, g)$, remains unpaired (the pairings – dashed edges – are labeled by their step number in the figure).

In Step 1, we vertically pair the nodes of the two longest solid-edge paths that are initiated by $\succsim^1$-nodes and do not have an already paired node. We pair $(\succsim^1, e)\&(\succsim^1, f)$, ..., $(\succsim^4, e)\&(\succsim^4, f)$. The final node $(\succsim^1, j)$ remains unpaired.

Continuing in a similar manner, we obtain the pairings in the figure.

Now we index each component in the figure (using both dashed edges and solid edges). There are seven of them.

We construct a new graph $\Gamma$ by simply representing each node pair by a new single node and each unpaired node by itself. Hence, we eliminate dashed edges. If there is a solid edge between two nodes of the old graph, we place an edge between the new nodes containing the old nodes. This new graph is represented in Figure 3.

As seen in Figure 3,[16] there are no cycles in this new graph. This is a sufficient condition to form the partition of $S$ so that C1 and C2 are satisfied. Each component of the old graph is now represented as a tree in the new graph. We choose its root as the leftmost node. Each tree can be partitioned independently from the other trees. We illustrate it on Tree 4 in the graph, as the others are trivial to partition. Tree 4 spans across 6 outcomes $d, e, f, i, j$.

We start with the root of the tree $(\succsim^9, d)$, an unpaired row, and arbitrarily assign it to

[16] *We dropped the preference profiles from the notation of the nodes and instead put them on the heading of the figure.*

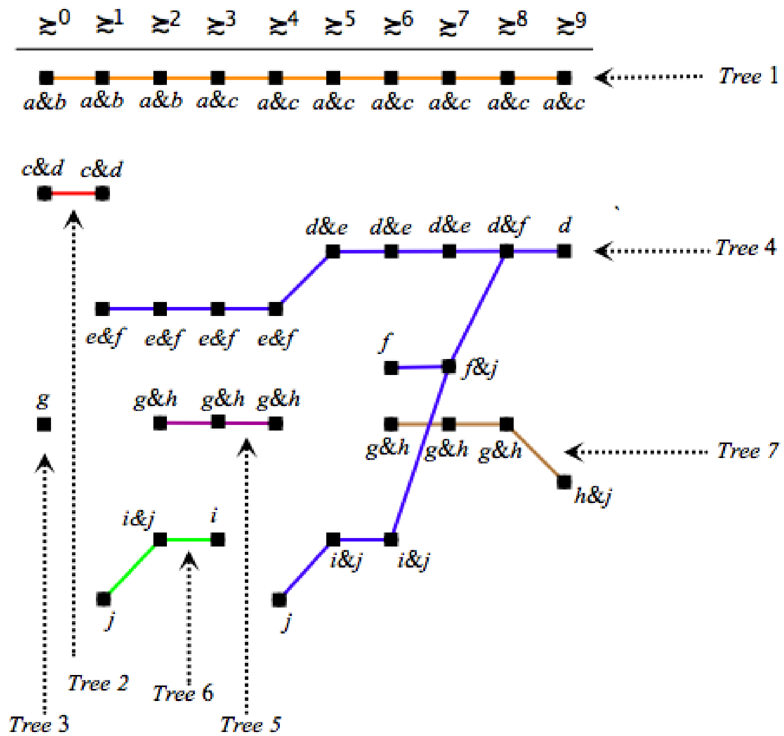


Figure 2: Graph Γ.

$S^+$. Then all rows $(\succsim^k, d)$ are assigned to $S^+$ for $k = 8, 7, 6, 5$. As the pair $(\succsim^8, d)\&(\succsim^8, f)$ is in the tree and $(\succsim^8, d)$ is already assigned to $S^+$, we assign $(\succsim^8, f)$ to $S^-$. Hence the rows on the branch stemming from $(\succsim^8, d)\&(\succsim^8, f)$ including $f$ should also be in $S^-$: $(\succsim^7, f), (\succsim^6, f)$ are assigned to $S^-$. Row $(\succsim^7, j)$ is assigned to $S^+$ as the pair $(\succsim^7, f)\&(\succsim^7, j)$ is in the tree and $(\succsim^7, f) \in S^-$. So are $(\succsim^k, j)$ for $k = 6, 5, 4$. Thus, rows $(\succsim^6, i)$ and $(\succsim^5, i)$ are assigned to $S^-$ as pairs $(\succsim^k, i)\&(\succsim^k, j)$ for $k = 6, 5$ are in the tree and each $(\succsim^k, j) \in S^+$.

We continue with the other branch of Tree 4 initiated by *pair* $(\succsim^8, d)\&(\succsim^8, f)$: Pairs $(\succsim^k, d)\&(\succsim^k, e)$ for $k = 7, 6, 5$ are in the tree. As $(\succsim^k, d) \in S^+$, we assign $(\succsim^k, e)$ to $S^-$. Hence all $e$ rows linked to the pair $(\succsim^5, d)\&(\succsim^5, e)$ consecutively need to be assigned to $S^-$. These are $(\succsim^k, e)$ for $k = 4, 3, 2, 1$. Finally, as pairs $(\succsim^k, e)\&(\succsim^k, f)$ for $k = 4, 3, 2, 1$ are in the tree and rows $(\succsim^k, e) \in S^-$ for $k = 4, 3, 2, 1$, then rows $(\succsim^k, f)$ for $k = 4, 3, 2, 1$ are assigned to $S^+$, concluding the partitioning of the rows associated with the nodes of Tree 4.

It is straightforward to verify that $S$'s Ghoulia-Houri difference through partition $\{S^+, S^-\}$ along each column of $A$ is $-1$, 0, or 1. $\diamond$

# 5 A Constructive Approach to Decomposition of Strategy-Proofness

In this section, we employ a constructive method to show the decomposability of strategy-proofness along with other desirable properties. We start by showing that strategy-proofness and feasibility together are not decomposable in general:

**Theorem 4** *For universal voting and house allocation problem models, strategy-proofness is not decomposable.*

**Proof of Theorem 4.** The proof is through a counterexample. Suppose we have a single agent $I = \{i\}$ and three outcomes $O = O_i = \{a, b, c\}$. With a single agent, feasibility constraints are the same for voting and house allocation.

Let $\phi$ be a random mechanism that assigns 1/2 probability to the agent's first choice and 1/2 probability to the agent's second choice for each preference the agent submits. As the first choice is given at least as high probability as lower choices by the mechanism, this mechanism is strategy-proof. On the other hand, it cannot be decomposed as a convex combination of feasible and strategy-proof deterministic mechanisms, as we prove below.

Suppose not. First suppose that there are $k$ deterministic mechanisms $\phi_1, \phi_2, ..., \phi_k$ that constitute a feasible strategy-proof decomposition of $\phi$, i.e., $\phi = \lambda_1\phi_1 + \lambda_2\phi_2 + .... + \lambda_k\phi_k$ for some probability distribution $\{\lambda_\ell\}$. Let's denote a preference relation of agent $i$ as $xyz$ meaning $x \succ_i y \succ_i z$ for all $\{x, y, z\} = \{a, b, c\}$. Feasibility is used implicitly in the below arguments:

1. When $\succsim_i = abc$, since $\phi(abc; i, a) = \frac{1}{2}$ and $\phi(abc; i, b) = \frac{1}{2}$, without loss of generality, let $\phi_1(abc; i, a) = ... = \phi_\ell(abc; i, a) = 1$ such that $\sum_{m=1}^{\ell} \lambda_m = \frac{1}{2}$ and $\phi_{\ell+1}(abc; i, b) = ... = \phi_k(abc; i, b) = 1$ such that $\sum_{m=\ell+1}^{k} \lambda_m = \frac{1}{2}$ for some $\ell$.

2. When $\succsim_i = acb$, by strategy-proofness of $\phi_s$ for all $s \in \{1, ..., \ell\}$ (from $abc$), $\phi_s(acb; i, a) = 1$, and hence, for all $m \in \{\ell + 1, ..., k\}$, $\phi_m(acb; i, c) = 1$.

3. When $\succsim_i = cba$, by strategy-proofness of $\phi_m$ for all $m \in \{\ell + 1, ..., k\}$ (from $acb$), $\phi_m(cba; i, c) = 1$, and hence, for all $s \in \{1, ..., \ell\}$, $\phi_s(cba; i, b) = 1$.

4. When $\succsim_i = bca$, by strategy-proofness of $\phi_s$ for all $s \in \{1, ..., \ell\}$ (from $cba$), $\phi_s(bca; i, b) = 1$, and hence, for all $m \in \{\ell + 1, ..., k\}$, $\phi_m(cba; i, c) = 1$.

5. When $\succsim_i = bca$, by strategy-proofness of $\phi_m$ for all $m \in \{\ell + 1, ..., k\}$ (from $abc$), $\phi_m(cba; i, b) = 1$, contradicting (4) above.

■

## 5.1 Strategy-proofness and Unanimity

In this subsection we use voting domains, i.e., for all $i, j \in I$, $o \in O \Rightarrow o_i = o_j$. As we deal with voting domains, by a slight abuse of notation we denote $O \equiv O_i$ for all $i \in I$ in this subsection. Single-peaked preferences are one of the cornerstones of voting theory, as they give rise to a wide range of strategy-proof voting mechanisms and negate the Gibbard-Satterthwaite impossibility result for universal domains.

Formally single-peaked preferences are defined as follows: Fix a linear order $\geq^*$ on $O$.[17] Let $\overline{o} = \max^* O$ and $\underline{o} = \min^* O$. We assume each agent $i \in I$ is endowed with a **single-peaked preference** $\succsim_i \in D_i$ with respect to this order; that is, there exists a **peak** outcome $o_i \in O$ such that for all $o <^* x \leq^* o_i$, $x \succ_i o$, and for all $o >^* x \geq^* o_i$, $x \succ_i o$. Let $o(\succsim_i) = o_i$ refer to the peak outcome of $\succsim_i$.

In addition to strategy-proofness, we use a weak but desirable efficiency property. A mechanism $\phi : D \to O$ is **unanimous** if, for any $\succsim \in D$, if there exists some $o \in O$ such that for all $i \in I$, $o_i \in Ch(O_i, \succsim_i)$, then for all $i \in I$, $\phi(\succsim; i, x_i) = 0$ for all $x_i \notin Ch(O_i, \succsim_i)$. It requires that whenever the preference profile is such that there exists a social outcome that can accommodate the top choice of each agent, then the mechanism does not select any social outcome that does not correspond to a top choice of each agent with a positive probability. For strict preferences this simply means that if there is a social choice that can accommodate the top choice of each agent then the mechanism chooses this social outcome with probability 1. For voting domains with strict preferences, such as the single-peaked domain, unanimity simply means that when each agent ranks the same candidate as his top choice, then this candidate is elected with probability 1.

Our main result in this section is as follows:

**Theorem 5** *Unanimity and strategy-proofness together are decomposable in the single-peaked preference voting model.*

Its proof follows directly from Lemma 2 and Theorem 6, below.

[17] Let $>^*$ be the associated strict ranking of alternatives. Let $\min^* X$ and $\max^* X$ refer to minimum and maximum of set $X \subseteq O$ under linear order $\geq^*$, respectively. Let $[x, y]^* = \{o \in O \mid x \leq^* o \leq^* y\}$ and $(x, y)^* = \{o \in O \mid x <^* o <^* y\}$ for all $x \leq^* y$. Let also $(x, y]^*$ and $[x, y)^*$ be appropriately defined. We will also use terminology such as median*, largest*, and smallest* referring to the respective unstarred terms applied to linear order $\geq^*$ .

## 5.2 Tops-onlyness of Strategy-proofness for Unanimous Random Mechanisms

It turns out that strategy-proof and unanimous random mechanisms are *tops-only* on the single-peaked voting domain, i.e., their outcome depends only on the peak choices reported by the agents. This was proved by Ehlers, Peters, and Storcken (2002) in their Proposition 5.2:

**Lemma 2 (Tops-onlyness of Strategy-proofness and Unanimity)** ***(Ehlers, Peters, and Storcken, 2002)*** *In a single-peaked preference voting model $(O, D)$ with linear order $\geq^*$ on $O$, if a random mechanism $\phi : \Delta D \to O$ is strategy-proof and unanimous, then its outcome depends only on top-ranked alternatives of agents.*

Using Lemma 2, we focus on *tops-only* mechanisms. We reformulate a **tops-only** mechanism as an indirect mechanism with a reduced message space: Each agent reports a top personal outcome and the mechanism $\psi$ maps this to a probability distribution over social outcomes, that is, $\psi : \times_{i\in I} O_i \to \Delta O$, where $\psi(o_1, ..., o_{|I|}; x)$ is the probability of $x \in O$ being chosen when agents $1, 2, ..., |I|$ report their top choices as $(o_1, ..., o_{|I|})$. Without loss of generality for a deterministic tops-only mechanism $\psi$, let $\psi(o_1, ..., o_{|I|}) \in O$ and let it denote the alternative that is chosen with probability 1 by $\psi$ when agents report $(o_1, ..., o_{|I|})$.

We say that a tops-only mechanism $\psi$ is **strategy-proof** if, for each agent, reporting his true top choice first-order stochastically dominates reporting any other alternative as top choice; that is, for all $i \in I$, $\succsim_i \in D_i$, $o_{-i} \in O^{|I|-1}$, $x_i \in O$, we have $\sum_{y \succsim_i z} \psi(o(\succsim_i), o_{-i}; y) \geq \sum_{y \succsim_i z} \psi(x_i, o_{-i}; y)$ for all $z \in O$.

The following seminal result of Moulin (1980) characterizes the full set of strategy-proof tops-only mechanisms on the single-peaked voting domain:

**Lemma 3** *(Moulin, 1980) On a single-peaked voting domain with linear order $\geq^*$ on $O$, a deterministic tops-only mechanism $\phi$ is strategy-proof if and only if there exists a vector of alternatives $\mathbf{p} = (p^S)_{S\subseteq I}$ such that $p^T \geq^* p^S$ for all $T \subseteq S \subseteq I$, and for all $o_1, .., o_{|I|} \in O^I$,*

$$\phi(o_1, .., o_{|I|}) = min^*_{S\subseteq I}\{max^*(S \cup \{p^S\})\}.$$

Thus, it can be observed that any deterministic strategy-proof tops-only mechanism $\phi$'s outcome is the median* of $|I|$ reported peaks and $|I|+1$ agent-specific constant alternatives. Let us relabel agents with respect to ranking of their reports so that we obtain $o_1 \leq^* o_2 \leq^* ... \leq^* o_{|I|}$. Whenever $o_i = o_j$ for some agents $i$ and $j$, given that agents $1, ..., k$ are relabeled, we relabel $i$ and $j$ so that $i$ gets a lower index number than $j$, if $p^{\{1,...,k,i\}} >^* p^{\{1,...,k,j\}}$, or $p^{\{1,...,k,i\}} = p^{\{1,...,k,j\}}$ and $i < j$. We also relabel the choices in vector $\mathbf{p} = (p^S)_{S \subseteq I}$ , respectively. Observe that, for any $T \subseteq I$ such that $o_\ell = \max^* T$, we have $\{1, ..., \ell\} \supseteq T$. Thus, by construction of $\mathbf{p}$, $p^{\{1,...,\ell\}} \leq^* p^T$. This implies $\min^*_{S \subseteq I : \max^* S = o_\ell}\{\max^*(S \cup \{p^S\})\} = \max^*\{o_\ell, p^{\{1,...,\ell\}}\}$. Hence,

$$\phi(o_1, .., o_{|I|}) = \min^*_{\ell \in \{0,...,|I|\}}\{\max^*\{o_\ell, p^{\{1,...,\ell\}}\}\}, \tag{7}$$

where $o_0 \equiv \underline{o}$. As $p^\emptyset \geq^* p^{\{1\}} \geq^* ... \geq^* p^{\{1,...,\ell\}} \geq^*, ..., \geq^* p^{\{1,...,|I|\}}$ and $o_1 \leq^* ... \leq^* o_{|I|}$, $\phi(o_1, .., o_{|I|})$ is the median* of these $2|I| + 1$ alternatives.

We refer to each $p^S$ as the **fixed ballot** for set $S \subseteq I$ and $\phi$ as the **generalized median voter rule** (**GMVR**) with respect to the fixed ballot vector $\mathbf{p} = (p^S)_{S \subseteq I}$. Let $m^{\mathbf{p}} : O^{|I|} \to O$ denote this mechanism from now on. Let $\mathcal{P} \subset 2^O$ be the set of all feasible fixed ballot vectors.

We prove the following result regarding unanimous mechanisms, which will help us use Lemma 3 in our analysis of the single-peaked domain.

**Lemma 4** *In a single-peaked preference voting model $(O, D)$ with linear order $\geq^*$ on $O$, a GMVR $m^{\mathbf{p}}$ is unanimous if and only if $p^\emptyset = \overline{o}$ and $p^I = \underline{o}$.*

**Proof of Lemma 4.** First, we show the necessity of this condition. Let $m^{\mathbf{p}}$ be a unanimous mechanism. Suppose, to the contrary of the claim, $x^\emptyset <^* \overline{o}$. Then for all $S \subseteq I$, $x^S <^* \overline{o}$. When all agents report $\overline{o}$, the outcome, $m^{\mathbf{p}}(\overline{o}, ..., \overline{o})$, is the median* of $|I| + 1$ alternatives smaller than $\overline{o}$ and $|I|$ alternatives equal to $\overline{o}$. The outcome is $x^\emptyset$, contradicting unanimity. Thus, $x^\emptyset = \overline{o}$. Similarly, suppose, to the contrary of the claim, $x^I >^* \underline{o}$. When all agents report $\underline{o}$, the outcome is $x^I$, contradicting unanimity. Thus, $x^I = \underline{o}$.

Next, we show the sufficiency of this condition. Let $m^{\mathbf{p}}$ be a GMVR such that $x^I = \underline{o}$ and $x^\emptyset = \overline{o}$. Let $o_i = y$ for all $i$ for some $y \in O$. Then as we can relabel the agents and obtain $m^{\mathbf{p}}(o_1, ..., o_{|I})$ as the median* of $x^{S_{|I|}}$,...,$x^{S_0}, o_1, ..., o_{|I|}$, for $S_\ell = \{1, ..., |I|\}$ for all $\ell \in \{1, ..., |I|\}$ and $S_0 = \emptyset$, $m^{\mathbf{p}}(o_1, ..., o_{|I}) = y$, showing that it is unanimous. ■

Thus, we have the following full characterization corollary to the above results regarding deterministic mechanisms:

**Corollary 1** *A deterministic mechanism is strategy-proof and unanimous in a single-peaked preference voting model $(O, D)$ with linear order $\geq^*$ on $O$ if and only if it is a GMVR with respect to a fixed ballot vector $\mathbf{p}$ such that $p^\emptyset = \overline{o}$ and $p^I = \underline{o}$.*

We continue with the following lemma, which pins down the most important property of the strategy-proof tops-only mechanisms that we use extensively in our characterization below in Theorem 6.

**Lemma 5** *For any strategy-proof tops-only mechanism $\phi$ in a single-peaked preference voting model $(O, D)$ with linear order $\geq^*$ on $O$, for all $i \in I$, $o_i \geq^* o'_i \in O$, $o_{-i} \in O^{|I|-1}$,*

$$\phi(o_i, o_{-i}; y) = \phi(o'_i, o_{-i}; y) \quad \forall y \in O \text{ such that } y >^* o_i \text{ or } y <^* o'_i.$$

**Proof of Lemma 5.** Suppose $\underline{o} \leq^* y <^* o'_i$. Let $w$ be the immediate successor of $y$ in the linear order $\geq^*$. In a preference relation peaked at $o'_i$, there can be any number of alternatives greater* than or equal to $o'_i$ that are strictly worse than $y$. Thus, by strategy-proofness of $\phi$, for all $w' >^* o'_i$, for all $o'_i$-peaked preferences $\succsim'_i$ such that $w'$ is the first choice greater* than $o'_i$ that is ranked just below $w$, we need to have

$$\begin{aligned}\sum_{z\in[w,o'_i]^*} \phi(o'_i, o_{-i}; z) + \sum_{z\in(o'_i,w')^*} \phi(o'_i, o_{-i}; z) &= \sum_{z \succsim'_i w} \phi(o'_i, o_{-i}; z) \\ &\geq \sum_{z \succsim'_i w} \phi(o_i, o_{-i}; z) \\ &= \sum_{z\in[w,o'_i]^*} \phi(o_i, o_{-i}; z) + \sum_{z\in(o'_i,w')^*} \phi(o_i, o_{-i}; z),\end{aligned}$$

where the first- and third-line equalities follow from single-peakedness of the preferences and the second-line inequality follows from the strategy-proofness of $\phi$.

If we pick $w'$ as the immediate successor of $o_i$ in the linear order $\geq^*$, then the above inequality reduces to

$$\sum_{z\in[w,o_i]^*} \phi(o'_i, o_{-i}; z) \geq \sum_{z\in[w,o_i]^*} \phi(o_i, o_{-i}; z). \tag{8}$$

Next, consider the $o_i$-peaked preference relation $\succsim_i$ where the immediate successor to $o_i$, $w'$, is ranked just below $w$. Then, by the strategy-proofness of $\phi$, we similarly have

$$\sum_{z\in[w,o_i]^*} \phi(o_i, o_{-i}; z) = \sum_{z \succsim_i w} \phi(o_i, o_{-i}; z) \geq \sum_{z \succsim_i w} \phi(o'_i, o_{-i}; z) = \sum_{z\in[w,o_i]^*} \phi(o'_i, o_{-i}; z), \tag{9}$$

where the first and last equalities follow from the single-peakedness of the preferences and the inequality follows from the strategy-proofness of $\phi$.

Inequalities 8 and 9 imply that

$$\sum_{z\in[w,o_i]^*} \phi(o_i, o_{-i}; z) = \sum_{z\in[w,o_i]^*} \phi(o'_i, o_{-i}; z). \tag{10}$$

The same equality is true for the alternative $y$, which is the immediate predecessor of $w$ in the linear order $\geq^*$. Thus,

$$\sum_{z\in[y,o_i]^*} \phi(o_i, o_{-i}; z) = \sum_{z\in[y,o_i]^*} \phi(o'_i, o_{-i}; z). \tag{11}$$

Subtracting the same-side terms from each other in Equalities 10 and 11 leads to

$$\phi(o_i, o_{-i}; y) = \phi(o'_i, o_{-i}; y).$$

A symmetric argument with the above proof for the case $\overline{o} \geq^* y >^* o_i$ shows that $\phi(o_i, o_{-i}; y) = \phi(o'_i, o_{-i}; y)$. ■

## 5.3 Decomposability of Tops-only Strategy-proofness

In this subsection, we show that strategy-proof tops-only mechanisms are decomposable on single-peaked domains. This result does not require unanimity as an additional property.

**Theorem 6** *Tops-only strategy-proofness is decomposable in a single-peaked preference voting model.*

**Proof of Theorem 6.** We will prove that for any tops-only strategy-proof mechanism $\phi$ in a single-peaked preference voting model $(O, D)$, there exist probabilities $(\alpha_{\mathbf{p}})_{\mathbf{p}\in\mathcal{P}}$ with (1) $\alpha_{\mathbf{p}} \in [0, 1]$ for all $(\alpha_{\mathbf{p}})_{\mathbf{p}\in\mathcal{P}}$, and (2) $\sum_{\mathbf{p}\in\mathcal{P}} \alpha_{\mathbf{p}} = 1$ such that for all $o_1, ..., o_{|I|} \in O$,

$$\phi(o_1, ..., o_{|I|}) = \sum_{\mathbf{p}\in\mathcal{P}} \alpha_{\mathbf{p}} m^{\mathbf{p}}(o_1, ..., o_{|I|}).$$

First, we define for all $S \subseteq I$, $\overline{o}_S = (o_i = \overline{o})_{i\in S}$ and $\underline{o}_S = (o_i = \underline{o})_{i\in S}$, i.e., the profiles of extreme peak reports of agents in set $S$.

We define implicitly the probability vector $\alpha = (\alpha_{\mathbf{p}})_{\mathbf{p}\in\mathcal{P}}$ by the following system of equations:

$$\sum_{\mathbf{p}\in\mathcal{P}:p^T=y} \alpha_{\mathbf{p}} = \phi(\underline{o}_T, \overline{o}_{-T}; y) \quad \forall\, y \in O,\ \forall T \subseteq I, \tag{12}$$

$$\alpha_{\mathbf{p}} \geq 0 \quad \forall\, \mathbf{p} \in \mathcal{P}.$$

If such an $\alpha$ exists, then fix $T \subseteq I$, we have $\sum_{\mathbf{p}\in\mathcal{P}} \alpha_{\mathbf{p}} = \sum_{y\in O} \sum_{\mathbf{p}\in\mathcal{P}:p^T=y} \alpha_{\mathbf{p}} = \sum_{y\in O} \phi(\underline{o}_T, \overline{o}_{-T}; y) = 1$, and hence, $\alpha$ defines a feasible lottery over GMVRs.

We first show that such a vector $\alpha$ exists.

***Claim 1.*** There is a solution $\alpha$ for Equation system 12.
***Proof.*** First observe that

$$\sum_{o\in[x,\overline{o}]^*} \phi(\underline{o}_S, \overline{o}_{-S}; o) \leq \sum_{o\in[x,\overline{o}]^*} \phi(\underline{o}_T, \overline{o}_{-T}; o) \quad \forall\, x \in O,\, \forall\, T \subsetneq S \subseteq I. \tag{13}$$

by the strategy-proofness of $\phi$ (repeatedly applied for agents in $S \setminus T$) and the single-peakedness of preferences. This is the most crucial property of a strategy-proof mechanism that will be used in our proof.

We claim that there exists some vector of fixed ballots $\mathbf{p}_1 \in \mathcal{P}$ (we use a subscript, as we construct a number of such vectors iteratively) such that $\phi(\underline{o}_T, \overline{o}_{-T}; p_1^T) > 0$ for all $T \subseteq I$. We inductively construct such a $\mathbf{p}_1 = (p_1^S)_{S\subseteq I}$ as follows:

- In the initial step: For $S = \emptyset$, set $p_1^S = \overline{o}$. Observe that by unanimity, $\phi(\underline{o}_\emptyset, \overline{o}_{-\emptyset}; \overline{o}) = 1 > 0$.

- As the inductive assumption, assume that for all sets $S \subseteq I$ such that $|S| \leq k < |I|$ for some fixed $k$, (1) we have constructed $x^S$ in a well-defined manner, (2) $\phi(\underline{o}_S, \overline{o}_{-S}; p_1^S) > 0$, and (3) $\phi(\underline{o}_S, \overline{o}_{-S}, x) = 0$ for all $x \in (p_1^S, \overline{o}]^*$.

  In the inductive step, take a set $S \subseteq I$ with $|S| = k+1$. Take the largest* $x$ such that $\phi(\underline{o}_S, \overline{o}_S; x) > 0$. Now by Equation 13, for any $T \subsetneq S$, $0 < \sum_{o\in[x,\overline{o}]} \phi(\underline{o}_S, \overline{o}_{-S}; o) \leq \sum_{o\in[x,\overline{o}]^*} \phi(\underline{o}_T, \overline{o}_{-T}; o)$. Thus, by the inductive assumption for construction of $p_1^T$, $p_1^T \geq^* x$. Let $p_1^S = x$.

Let $\Phi^1 = [\phi(\underline{o}_S, \overline{o}_{-S}; x)]_{S\subseteq I, x\in O}$ be the $2^{|I|} \times |O|$ dimensional matrix represented by the mechanism $\phi$'s outcomes at extreme peaks. Let $M^1$ be a matrix with the same dimensions representing the GMVR $m^{\mathbf{p}_1}$ such that for all $T, x$, $M^1(T, x) = m^{\mathbf{p}_1}(\underline{o}_T, \overline{o}_{-T}, x)$. Observe that $m^{\mathbf{p}_1}(\underline{o}_T, \overline{o}_{-T}, x) = \begin{cases} 1 & \text{if } x = p_1^T \\ 0 & \text{otherwise} \end{cases}$, which follows directly from the definition of $m^{\mathbf{p}_1}$, i.e., the median* of reported peaks $(\underline{o}_T, \overline{o}_{-T})$ and the required fixed ballots in vector $\mathbf{p}_1$ is always equal to $p_1^T$.

Let $T^1 \in \arg\min_{T\subseteq I} \phi(\underline{o}_T, \overline{o}_{-T}, p_1^T)$.
Define

$$\alpha_{\mathbf{p}_1} = \phi(\underline{o}_{T^1}, \overline{o}_{-T^1}; p_1^{T^1}) \equiv \Phi^1(T^1, p_1^{T^1}),$$

i.e. as the smallest entry in $\Phi^1$ for the cells $(T, p_1^T)$ for all $T \subseteq I$. Observe that this is positive by construction of $\mathbf{p}_1$.

Form matrix

$$\Phi^2 = \frac{\Phi^1 - \Phi^1(T^1, p_1^{T^1})M^1}{1 - \Phi^1(T^1, p_1^{T^1})}. \tag{14}$$

Observe that either (1) $\Phi^2$ is a zero matrix, or (2) $\Phi^2$ is non-negative and $\sum_{x \in O} \Phi^2(T, x) = 1$ for all $T \subseteq O$.

In Case (1), we are done with the construction of $\alpha$. In Case (2), we define a random mechanism $\phi^2$ such that $\phi^2(\underline{o}_T, \overline{o}_{-T}; x) = \Phi^2(T, x)$ for all $T \subseteq I, x \in O$. Observe that $\Phi^2$ has at least one more zero than $\Phi^1$. Moreover, if the property in Equation 13 holds for $\phi^2$, we can replicate the above procedure for $\Phi^2$ and obtain a new generalized median voter voter rule with a vector of fixed ballots $\mathbf{p}_2$.

Thus, we prove the property in Equation 13 holds for $\phi^2$:

Pick $T \subsetneq T' \subseteq I$ and $x \in O$.

We have $p_1^{T'} \leq^* p_1^T$, $m^{\mathbf{p}_1}(\underline{o}_{T'}, \overline{o}_{-T'}; p_1^{T'}) = 1$, and $m^{\mathbf{p}_1}(\underline{o}_T, \overline{o}_{-T}; p_1^T) = 1$ by definition. Thus,

$$\sum_{o \in [x, \overline{o}]^*} m^{\mathbf{p}_1}(\underline{o}_{T'}, \overline{o}_{-T'}; o) - \sum_{o \in [x, \overline{o}]^*} m^{\mathbf{p}_1}(\underline{o}_T, \overline{o}_{-T}; o) = \begin{cases} 0 & \text{if } x \leq^* p_1^{T'} \text{ or } x >^* p_1^T \\ -1 & \text{otherwise} \end{cases}. \tag{15}$$

Also observe that by Equation 13 (for $\phi$) we have for all $x$, $\sum_{o \in [x, \overline{o}]} \phi(\underline{o}_{T'}, \overline{o}_{-T'}; o) - \sum_{o \in [x, \overline{o}]} \phi(\underline{o}_T, \overline{o}_{-T}; o) \leq 0$. Also we have $\phi(\underline{o}_T, \overline{o}_{-T}, p^T) \geq \alpha_{\mathbf{p}_1}$ by definition of $\alpha_{\mathbf{p}_1}$ and $\phi(\underline{o}_{T'}, \overline{o}_{-T'}, x) = 0$ for all $x >^* p_1^{T'}$ by inductive assumption for the construction of $\mathbf{p}_1$. Hence, for all $x \in (p_1^{T'}, p_1^T]^*$,

$$\sum_{o \in [x, \overline{o}]^*} \phi(\underline{o}_{T'}, \overline{o}_{-T'}; o) - \sum_{o \in [x, \overline{o}]^*} \phi(\underline{o}_T, \overline{o}_{-T}; o) \leq -\alpha_{\mathbf{p}_1}.$$

Thus, we showed that

$$\sum_{o \in [x, \overline{o}]^*} \phi(\underline{o}_{T'}, \overline{o}_{-T'}; o) - \sum_{o \in [x, \overline{o}]^*} \phi(\underline{o}_T, \overline{o}_{-T}; o) \leq \begin{cases} 0 & \text{if } x \leq^* p_1^{T'} \text{ or } x >^* p_1^T \\ -\alpha_{\mathbf{p}_1} & \text{otherwise} \end{cases}. \tag{16}$$

Equations 15 and 16 and the definition of $\Phi^2$ in Equation 14 imply

$$\sum_{o \in [x, \overline{o}]^*} \phi^2(\underline{o}_{T'}, \overline{o}_{-T'}; o) - \sum_{o \in [x, \overline{o}]^*} \phi^2(\underline{o}_T, \overline{o}_{-T}; o) \leq 0.$$

Iterating the above procedure, we find fixed ballot vectors $\mathbf{p}_1, ..., \mathbf{p}_{l^*}$ and corresponding matrices for the GMVRs $M^1, ..., M^{l^*}$, non-negative probability matrices $\Phi^1$,...,$\Phi^{l^*+1}$ satisfying

$$\Phi^{l+1} = \frac{\Phi^l - \Phi^l(T^l, p_l^{T^l})M^l}{1 - \Phi^l(T^l, p_l^{T^l})}$$

for each $l \leq l^*$ and $\Phi^{l^*+1}$ is the zero matrix, and weights

$$\alpha_{\mathbf{p}_l} = \prod_{l'=1}^{l-1} \left(1 - \Phi^{l'}(T^{l'}, p_{l'}^{T^{l'}})\right) \Phi^l(T^l, p_l^{T^l})$$

where $T^l \in \arg\min_{T \subseteq I} \Phi^l(T, p_l^T)$ for all $l \leq l^*$.

As $\Phi^l(T, p_l^T) > 0$ for all $l \leq l^*$ and $T \subseteq I$ by construction, $1 > \alpha_{\mathbf{p}_l} > 0$ for all $l \leq l^*$ if $l^* > 1$, and $\alpha_{\mathbf{p}^1} = 1$ if $l^* = 1$.

An $l^*$ such that $\Phi^{l^*+1}$ is the zero matrix exists, because each $\Phi^l$ matrix has at least one more zero cell than $\Phi^{l-1}$ matrix has. We conclude by assigning $\alpha_{\mathbf{p}} = 0$ for all $\mathbf{p} \in \mathcal{P} \setminus \{\mathbf{p}_1, ..., \mathbf{p}_{l^*}\}$ (i.e., the ones not encountered in the above process).$\diamond$

Let mechanism $\rho$ be defined as

$$\rho(o_1, ..., o_{|I|}) = \sum_{\mathbf{p} \in \mathcal{P}} \alpha_{\mathbf{p}} m^{\mathbf{p}}(o_1, ..., o_{|I|}).$$

We show that for all $y \in O$

$$\rho(o_1, ..., o_{|I|}; y) = \phi(o_1, ..., o_{|I|}; y).$$

Fix $y \in O$. We relabel agents $1, ..., |I|$ such that $o_1 \leq^* o_2 \leq^* ... \leq^* o_{|I|}$. Moreover, we relabel fixed ballot sets $\mathbf{p} \in \mathcal{P}$ and their probabilities $\alpha_{\mathbf{p}}$ accordingly.

Let

- $S_0 = \emptyset$,
- $S_\ell = \{1, ..., \ell\}$ for all $\ell \in I$,
- $o_0 = \underline{o}$,
- $o_{|I|+1} = \overline{o}$.

As $\rho(o_1, ..., o_{|I|}; y)$ is the probability of alternative $y$ being chosen at lottery $\rho(o_1, ..., o_{|I|})$, we need to count the cases when $y$ is chosen under a GMVR. Observe that any $z \in O$

is chosen by a GMVR for some fixed ballots when agents report $o_1, ..., o_{|I|}$, for example, when $p^S = z$ for all $S$. Whenever $y$ is chosen for some fixed ballots $\mathbf{p}$, by Equation 7, we have $y = \max^*\{o_k, p^{S_k}\}$ for some $k \in \{0, ..., |I|\}$. If there are multiple such $k$'s, we choose the maximum index $k$ such that $y = \max^*\{o_k, p^{S_k}\}$ is greater* than or equal to the $|I| + 1$ alternatives $o_1, ..., o_k, p^{S_{|I|}}, ..., p^{S_k}$ and less* than or equal to the $|I|$ alternatives $o_{k+1}, ..., o_{|I|}, p^{S_{k-1}}, ..., p^{S_0}$ (i.e., $o_k$ or $p^{S_k}$ is the median* with the maximum possible index $k$). This choice of $k$ immediately rules out a case like $o_k <^* y = o_{k+1}$. In this case, $p^{S_{k+1}} \leq^* p^{S_k} = y = o_{k+1}$. But then, $o_{k+1} \geq^* o_1, ..., o_{k+1}, p^{S_{|I|}}, ..., p^{S_{k+1}}$ and $o_{k+1} \leq^* o_{k+2}, ..., o_{|I|}, p^{S_k}, ..., p^{S_0}$; hence $k+1$ should have been chosen as $k$ in the first place, leading to a contradiction to the maximality of $k$. Thus, we have either $o_k <^* y <^* o_{k+1}$, or $o_k = y <^* o_{k+1}$, or $o_k = y = o_{k+1}$. We deal with the first case and part of the second case below:

***Claim 2.*** If either $o_k <^* y <^* o_{k+1}$, or $y = \underline{o} <^* o_{k+1}$, or $o_k <^* \overline{o} = y$, then $\phi(o_1, ..., o_{|I|}; y) = \rho(o_1, ..., o_{|I|}; y)$.

***Proof.*** Observe that, in this case, $k$ is independent of the choice of fixed ballots $\mathbf{p}$ as long as they result in $m^{\mathbf{p}}(o_1, ..., o_{|I|}) = y$. Moreover, we have $y = p^{S_k}$ for each such $\mathbf{p}$. Also recall that for each such $\mathbf{p}$, as $p^{S_k} \geq^* p^{S_\ell}$ for all $\ell > k$ and $y$ is the minimum of all $\max^*\{o_\ell, p^{S_\ell}\}$, then $o_\ell \geq^* y$ for all $\ell > k$. Hence, for all $o \in \{o_1, ..., o_k, p^{S_{|I|}}, p^{S_{|I|-1}}, ..., p^{S_{k+1}}\}$, $o \leq^* y$, and for all $o \in \{o_{k+1}, ..., o_{|I|}, p^{S_0}, ..., p^{S_{k-1}}\}$, $o \geq^* y$ and $y = p^{S_k}$.

We rewrite $\rho(o_1, ..., o_{|I|}; y)$ as follows:

$$\rho(o_1, ..., o_{|I|}; y) = \sum_{\mathbf{p} \in \mathcal{P}:\, p^{S_k} = y} \alpha_{\mathbf{p}} = \phi(\underline{o}_{S_k}, \overline{o}_{-S_k}; y) = \phi(o_1, ..., o_{|I|}; y).$$

where the second equality follows from the implicit definition of $\alpha$ in Equation system 12, and the last equality follows from

1. if $o_k <^* y <^* o_{k+1}$, through Lemma 5 applied separately for $S_k$ and $I \setminus S_k$;

2. if $y = \underline{o} <^* o_{k+1}$, through $o_1 = ... = o_k = \underline{o}$ and Lemma 5 applied for $I \setminus S_k$; and

3. if $o_k <^* y = \overline{o}$, through $o_{k+1} = ... = o_{|I|} = \overline{o}$ and Lemma 5 applied for $S_k$.

◇

The below claim deals with the case when $o_k = y$ and some other reports are possibly equal to $y$:

***Claim 3.*** If either

- $o_{j-1} <^* o_j = ... = o_k = ... = o_\ell = y <^* o_{\ell+1}$ for some $j, \ell$ such that $0 < j \leq k \leq \ell < |I| + 1$, or
- $\underline{o} = o_1 = ... = o_k = ... = o_\ell = y <^* o_{\ell+1}$ for some $\ell$ such that $k \leq \ell < |I| + 1$, or
- $o_{j-1} <^* o_j = ... = o_k = ... = o_{|I|} = y = \overline{o}$ for some $j$ such that $0 < j \leq k$,

then $\phi(o_1, ..., o_{|I|}; y) = \rho(o_1, ..., o_{|I|}; y)$.

***Proof.*** If $y = \underline{o}$, let $j = 1$ and if $y = \overline{o}$, let $\ell = |I|$. Observe that by Claim 2, for all $z \neq y$,

$$\phi(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; z) = \rho(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; z).$$

As both $\phi(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell})$ and $\rho(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell})$ are probability distributions, the above equality implies that

$$\begin{aligned}\phi(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; y) &= 1 - \sum_{z \neq y} \phi(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; z) \\ &= 1 - \sum_{z \neq y} \rho(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; z) \\ &= \rho(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; y).\end{aligned}$$

Moreover, as both $\rho$ and $\phi$ are strategy-proof, Lemma 5 used for $S_{j-1}$ and $I \setminus S_\ell$ gives us

$$\phi(o_1, ..., o_{|I|}; y) = \phi(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; y) = \rho(\underline{o}_{S_{j-1}}, o_j, ..., o_\ell, \overline{o}_{-S_\ell}; y) = \rho(o_1, ..., o_{|I|}; y).$$

$\diamond$

Claims 2 and 3 conclude the proof of the theorem. ■

## 5.4 Strategy-proofness, Unanimity, and Anonymity

Another desirable property of a random mechanism is *anonymity*, the outcome of the mechanism being dependent only on the set of reported preferences but not on who reported them. Formally, a mechanism $\phi : D \to \Delta O$ is **anonymous** on a symmetric domain such that $D_i = D_j$ for all $i, j \in I$ if, for all $i, j \in I$ and $\succsim \in D$, $\phi(\succsim) = \phi(\succsim^{i \leftrightarrow j})$ where $\succsim^{i \leftrightarrow j}$ is the preference profile obtained from $\succsim$ by swapping $i$'s preferences with $j$'s preferences. On the single-peaked domain, the implication of anonymity in addition to strategy-proofness and unanimity is such that every deterministic mechanism with these properties is a (tops-only) GMVR $m^{\mathbf{p}}$ with $p^\emptyset = \overline{o}$ and $p^I = \underline{o}$ and $p^T = p^S$ for all $S, T \subseteq I$ with $|S| = |T|$.

Observe that if $\phi$ is a random mechanism that is unanimous, anonymous, and strategy-proof on a single-peaked domain, then in our proof of Theorem 6 in the proof of Claim 1, every GMVR constructed that has a positive weight in the decomposition of $\phi$ is also anonymous. Thus, the following theorem is a corollary to the proof of Theorem 6 and Lemma 2:

**Theorem 7** *Anonymous, unanimous, and strategy-proofness together are decomposable in a single-peaked preference voting model.*

# 6 Conclusion

In this paper, we study which desirable properties of a random mechanism survive decomposition of the mechanism as a lottery over deterministic mechanisms that also hold such properties. When desirable properties survive decomposition, we can focus our mechanism design efforts on deterministic mechanisms. We represented properties of random mechanisms as linear constraints, and, using combinatorial integer programming, we studied a sufficient condition, the total unimodularity of the constraint, for decomposability of such linear constraints. Examples of such totally unimodular constraints are various feasibility constraints, individual rationality, and strategy-proofness in certain individual choice models. On the other hand, strategy-proofness is not decomposable in general. Moreover, strategy-proofness, unanimity, and feasibility are not totally unimodular in many collective choice models. In such models, to decompose strategy-proofness, we employed other methods. Using a direct constructive approach, we proved that feasibility, strategy-proofness, and unanimity are decomposable on the non-dictatorial single-peaked voting domains. The proof of this result also shows that feasibility, strategy-proofness, anonymity, and unanimity are decomposable on the same domains.

It is well known that in many settings, random mechanisms can do better than deterministic ones, while in other settings, deterministic mechanisms do as well as random ones. One question for future research is to identify the connection of the latter domains and various combinatorial properties of constraints that are being decomposed. Total unimodularity is one such sufficient condition, although it could be rather too strong for some constraints, such as strategy-proofness in some settings. Another question is which constraints are totally unimodular and which ones are not, as total unimodularity is a relatively well-understood concept in integer programming theory.

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