Swaps on Networks

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Abstract

A group of agents exchange discrete resources on a network without recourse to monetary transfers. Allowing for an arbitrary network structure, we show that there is a unique core outcome in the exchange problem. This unique outcome may be implemented via a natural extension of Gale's Top Trading Cycle mechanism, which is shown to be the unique mechanism that is individually strategy-proof, Pareto efficient, and individually rational.

1 Introduction

Suppose each of a finite group of agents is endowed with an object (a discrete resource). Each agent can ship his object to some other agents: the network of shipment possibilities is determined by technological constraints such as the perishability of the objects and shipment costs. Agents have preferences over objects that can be shipped to them. When monetary payments are not possible, how do the agents exchange objects?

Allowing for an arbitrary network structure, we show that there is a unique core outcome in the exchange problem. This unique outcome may be implemented via a natural extension of Gale's Top Trading Cycle mechanism (first reported in Shapley and Scarf (1974)) to the network setting. Furthermore, Top Trading Cycles is the unique mechanism that is individually strategy-proof, Pareto efficient, and individually rational.

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The uniqueness of the core outcome, and the individual strategy-proofness, Pareto efficiency, and individual rationality of Top Trading Cycles directly build on the insights of Roth and Postlewaite (1977) and Roth (1982), who established the analogues of these results in the standard house exchange problem of Shapley and Scarf (1974).[1] The claim that Top Trading Cycles is the unique mechanism satisfying these criteria is more subtle. For the standard house exchange model, identical to the special case of our setting in which any two agents can swap the objects (complete network), uniqueness was shown by Ma (1994). His result relies on the assumption that all profiles of preferences are possible and this assumption fails for most exchange networks, e.g. an agent cannot report as her top choice an object that cannot be shipped to her via the network.[2]

To the best of my knowledge, this is the first analysis of strategy-proof exchange on networks in a no-transfer environment.[3] Exchange on networks with transfers has been analyzed by, for instance, Bochet and Ilkilic (2015), Bramoullé, Kranton, and D'Amours (2014), and Manea (2015). The closest to our analysis among these papers is the work of Bochet and Ilkilic, who analyzed efficient, strategy-proof, individually rational, and pairwise stable exchange with transfers. The present paper is different both because it looks at a different setting (no-transfers), but also because the present analysis does not hinge on any stability assumptions. In the no-transfer environment, Schummer and Vohra (2002) analyzed the choice of location on a network when agents have single-peaked preferences.

2 Model

Let I denote the set of agents. Each agent possesses an indivisible object. We denote by $h(i)$ the endowment of agent $i \in I$ and by $H = \{h(i) \,|i \in I\}$ the set of all objects. Let $N \subseteq I \times I$ be the network of shipment possibilities: we say that an object of agent i can be shipped to agent j if and only if $(i, j) \in N$. We assume that each agent is connected to himself or herself, $(i, i) \in N$ for all $i \in I$. Agents have strict preferences over objects that can be shipped to them. We denote by $\mathcal{P}_i$ the set of preferences of agent i, and by $\mathcal{P} = \times_{i \in I} \mathcal{P}_i$ the set of all preference profiles.

One special case of this framework is that the objects can be shipped between any two agents, $N = I \times I$; this case corresponds to the standard Shapley and Scarf house exchange

[1] Indeed, given a fixed profile of agents' preferences, the exchange of resources on a network can be interpreted as a standard house exchange problem in which an agent to whom an object cannot be shipped is assumed to find such an object unacceptable. The standard house exchange problem corresponds to exchange on a complete network.

[2] Ma's result has been extended by Pycia and Ünver (2016) and Tang and Zhang (2015) to richer single-unit demand models without the network structure, and by Pápai (2007) to multi-unit demand models without network structure.

[3] Jun Zhang (private communication) independently extended Top Trading Cycles to network problems; his extension is not strategy-proof but it satisfies other desirable properties.

model. Another case of this framework obtains if we assume that there is an underlying network of connections between agents $\tilde{N} \subseteq I \times I$ and that an object can only be transported to a direct neighbor in $\tilde{N}$. Yet another example obtains when there is an underlying network of connections and each object can be shipped between neighbors on the network and between any two agents who share a neighbor. Any transportation costs—whether heterogenous or homogenous—are reflected in agents' preferences over objects. We allow asymmetric networks, in particular we allow the situation in which agent i might posses a non-perishable object that he can ship to agent j while agent j posses a perishable object that cannot be shipped to agent i.

An outcome is a mapping from agents to the objects they receive, $\mu : I \to H$. We restrict attention to outcomes that are feasible, that is, if $\mu(i) = h(j)$ then $(i, j) \in N$. A mechanism is a mapping from preference profiles to outcomes.

The extension of Top Trading Cycles (TTC) to this setting is straightforward. The mechanism runs in rounds. In each round, each agent points to their most preferred object. There is at least one exchange cycle, and all objects along all cycles are swapped. This procedure is equivalent to the standard TTC in an auxiliary house exchange problem in which objects can be shipped between any two agents but each agent to whom an object cannot be shipped in network N is assumed to find such an object unacceptable.

The core, individual and group strategy-proofness, Pareto efficiency, and individually rationality are defined in the standard way. An allocation μ is a **core** allocation if there is no group of agents $J \subseteq I$ and an allocation ν such that $\nu(J) = h(J)$, $\nu(i) \succeq_i \mu(i)$ for all $i \in J$, and $\nu(i) \succ_i \mu(i)$ for some $i \in I$.[4] An allocation μ is Pareto efficient if no other allocation makes each agent weakly better off, and at least one agent strictly better off, that is there exists no other allocation ν such that for all $i \in I$, $\nu(i) \succeq_i \mu(i)$, and for some $i \in I$, $\nu(i) \succ_i \mu(i)$. A mechanism is **Pareto efficient** if it assigns a Pareto-efficient matching to every preference profile. A mechanism φ is **individually strategy-proof** if truthful revelation of preferences is a weakly dominant strategy for any agent, that is for all $\succ \in \mathbf{P}$, there is no $i \in I$ and $\succ'_i \in \mathbf{P}_i$ such that $\varphi[\succ'_i, \succ_{-i}](i) \succ_i \varphi[\succ](i)$. A mechanism φ is **group strategy-proof** if no group of agents can misreport their preferences in such a way that each agent in the group gets a weakly better object, and at least one agent in the group gets a strictly better object irrespective of the preference reports of the agents not in the group, that is for all $\succ \in \mathbf{P}$, there exists no $J \subseteq I$ and $\succ'_J \in \mathbf{P}_J$ such that $\varphi[\succ'_J, \succ_{-J}](i) \succeq_i \varphi[\succ](i)$ for all $i \in J$, and $\varphi[\succ'_J, \succ_{-J}](j) \succ_j \varphi[\succ](j)$ for at least one $j \in J$. A mechanism is **individually rational** if it always selects an individually rational outcome; an outcome is individually rational if it assigns each agent an object that is at least as good as this agent's endowment: $\mu(i) \succeq_i h(i)$ for all $i \in I$.

[4]Because agents have strict preferences over objects, this notion of the core is equivalent to a seemingly weaker notion in which an allocation μ is a core allocation if there is no group of agents $J \subseteq I$ and an allocation ν such that $\nu(J) = h(J)$ and $\nu(i) \succ_i \mu(i)$ for all $i \in J$.

3 Main Result

Our main result is as follows:

Theorem 1. *For any network of shipment possibilities N: The outcome of Top Trading Cycles is the unique core outcome, the Top-Trading-Cycles mechanism is group strategy-proof, and it is the unique mechanism that is strategy-proof, individually rational, and Pareto efficient.*

Proof. Fixing a profile of agents' preferences, the exchange of resources on a network can be interpreted as a standard house exchange problem in which an agent to whom an object cannot be shipped is assumed to find such an object unacceptable; our claim regarding the core then follows from Roth and Postlewaite (1977). The same argument shows that TTC is individually rational and Pareto efficient.[5] To see that it is also group strategy-proof notice that allowed deviations in our problem are a subset of deviations in the associated standard house exchange problem; hence the group strategy-proofness in our setting follows from the group strategy-proofness in the standard house exchange, where it was shown by Roth (1982). In particular, TTC is individually strategy-proof.

It remains to prove that any mechanism ϕ satisfying the three conditions in the theorem is equivalent to the Top-Trading-Cycles mechanism, denoted ψ.[6] For $J \subseteq I$ and $\ell = 0, 1, ..., |J|$, let $\mathcal{P}_\ell(J)$ be the set of preference profiles in which at least $|J| - \ell$ agents in J satisfy the following property (*): there does not exist an object h' such that $\psi(\succ)(i) \succ h' \succ h(i)$.

The proof is by recursion on ℓ and the rounds of ψ. Let I_k be the set of agents who receive objects in the k-th round of ψ. Suppose $\succ \in \mathcal{P}_0(I_1)$ and consider $i \in I_1$. If $\varphi(\succ)(i) \neq \psi(\succ)(i)$ then the individual rationality of ϕ implies that $\varphi(\succ)(i) = h(i)$. Thus, under ψ agent i is matched in a cycle with at least one other agent; in this cycle each agent j receives their top choice, and he strictly prefers this choice over $h(j)$. Under φ, the individual rationality and $\succ \in \mathcal{P}_0$ implies that each agent j in this cycle receives $h(j)$. This contradicts Pareto efficiency, proving that $\varphi(\succ)(i) = \psi(\succ)(i)$ for $i \in I_1$ and $\succ \in \mathcal{P}_0(I_1)$.

To run an induction on ℓ within the first round, let $\ell \geq 0$, suppose that $\varphi(\succ)(i) = \psi(\succ)(i)$ for $i \in I_1$ and $\succ \in \mathcal{P}_\ell(I_1)$, and take $\succ \in \mathcal{P}_{\ell+1}(I_1)$. First, suppose $i \in I_1$ and $\succ_i$ fails (*). Let $\succ'$ rank objects in the same way as $\succ$ except that i moves up object $h(i)$ in his ranking so that property (*) is satisfied. By construction, $\succ' \in \mathcal{P}_\ell(I_1)$ and by the inductive assumption, $\varphi(\succ')(j) = \psi(\succ')(j)$ for all $j \in I_1$. Because ψ is group strategy-proof, we also have $\psi(\succ')(j) = \psi(\succ)(j)$ for all $j \in I$. Because $\varphi(\succ')(i) = \psi(\succ')(i)$ is the top choice of i, the individual strategy-proofness of φ implies that $\varphi(\succ)(i) = \varphi(\succ')(i) = \psi(\succ)(i)$. We have thus shown

[5] Alternatively, we can prove these statements using a direct recursive argument using the notation introduced in the next paragraph. For instance, notice that if an allocation μ is in the core then $\mu(i) = \psi(\succ)(i)$ for agents i matched in the first round of TTC ψ, and so on recursively.

[6] In markets without the network structure, the analogue of this claim was proven by Ma (1994). Since this result is more subtle and relies on an interplay between different preference profiles, there is no direct way to derive our network result from Ma's seminal theorem.

that $\varphi(\succ)(i) = \psi(\succ)(i)$ for agents $i \in I_1$ whose preference ranking fails (*). Second, suppose $i \in I_1$ has a preference ranking that satisfies (*) then either all agents in the cycle of i under ψ have preference rankings satisfying (*), or there is an agent that does not. In the former case, the argument we used for $\succ \in \mathcal{P}_0$ shows that $\varphi(\succ)(i) = \psi(\succ)(i)$. In the latter case, by way of contradiction suppose the claim is false. Then there exists an agent j whose ranking satisfies (*), $\varphi(\succ)(j) \neq \psi(\succ)(j)$, and whose object under ψ is allocated to an agent j' such that $\varphi(\succ)(j') = \psi(\succ)(j')$. But then (*) and individual rationality imply that $\varphi(\succ)(j) = \psi(\succ)(j)$; a contradiction.

We have thus shown that $\varphi(\succ)(i) = \psi(\succ)(i)$ for all $\succ$ and $i \in I_1$. To run an induction on the rounds of ψ, suppose the equality is true for all $\succ$ and $i \in I_1 \cup ... \cup I_r$ for some $r \geq 1$. The argument above can then be used to show that $\varphi(\succ)(i) = \psi(\succ)(i)$ for all $\succ$ and $i \in I_1 \cup ... \cup I_{r+1}$. The only change is that now we look at agents from I_{r+1} who obtain their most-preferred objects not matched before round $r+1$ (as opposed to their most-preferred objects) under ψ and who obtain at best these objects under φ. **QED**

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