ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 80, No. 1 (January, 2012), 323–362

STABILITY AND PREFERENCE ALIGNMENT IN MATCHING AND COALITION FORMATION

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Econometrica, Vol. 80, No. 1 (January, 2012), 323–362

STABILITY AND PREFERENCE ALIGNMENT IN MATCHING AND COALITION FORMATION

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We study matching and coalition formation environments allowing complementarities and peer effects. Agents have preferences over coalitions, and these preferences vary with an underlying, and commonly known, state of nature. Assuming that there is substantial variability of preferences across states of nature, we show that there exists a core stable coalition structure in every state if and only if agents' preferences are pairwise-aligned in every state. This implies that there is a stable coalition structure if agents' preferences are generated by Nash bargaining over coalitional outputs. We further show that all stability-inducing rules for sharing outputs can be represented by a profile of agents' bargaining functions and that agents match assortatively with respect to these bargaining functions. This framework allows us to show how complementarities and peer effects overturn well known comparative statics of many-to-one matching.

KEYWORDS: Many-to-one matching, assortative matching, coalition formation, stability, core, complementarities, peer effects, sharing rules, consistency, Nash bargaining.

1. INTRODUCTION

AGENTS FORM COALITIONS in many environments. In some of them, two distinct groups of agents match many to one: colleges form coalitions with incoming students, and hospitals form coalitions with new residents. In others, the structure is one-sided: individuals form private clubs and partnerships, and firms form business alliances. What coalitions will form in these environments? A natural answer relies on the notion of stability: a partition of agents into coalitions is (core) stable if there does not exist a counterfactual coalition which its members prefer to their coalitions in the partition.[2]

The present paper introduces a unified framework to study coalition formation including many-to-one matching and one-sided coalition formation. The

[1]Sections 4, 5.1, and 6.2 of this paper synthesize the first two chapters of my dissertation at MIT. I would like to thank Bengt Holmström, Glenn Ellison, Haluk Ergin, Robert Gibbons, Anna Myjak-Pycia, Michael Piore, and Jean Tirole for their advice and support. Sections 3, 4, and 7 were developed while I was at Penn State. For their generous comments, I am grateful to various seminar audiences, Andrew Atkeson, Abhijit Banerjee, Simon Board, Peter Chen, Dora Costa, Federico Echenique, Michael Egesdal, Alfred Galichon, Edward Green, Christian Hellwig, Hugo Hopenhayn, Sergei Izmalkov, George Mailath, Moritz Meyer-ter-Vehn, Kenneth Mirkin, Benny Moldovanu, Hervé Moulin, Ichiro Obara, Marco Ottaviani, Alvin Roth, Mordechai Schwarz, Alex Teytelboym, William Thomson, Hannu Vartiainen, Rakesh Vohra, Birger Wernerfelt, and, especially, Jingyi Xue and William Zame. The constructive comments of the editor and four referees greatly improved the paper. Financial support from the Hausdorff Institute in Bonn, MIT Industrial Performance Center, and Koźmiński University in Warsaw is gratefully acknowledged.

[2]See Gale and Shapley (1962), Buchanan (1965), and Farrell and Scotchmer (1988). Roth (1984) and Roth and Peranson (1999) linked the lack of stability to market failures.

 DOI: 10.3982/ECTA7143

main results establish a necessary and sufficient condition for stability, taking agents' ordinal preferences as primitives. We apply this tight condition for stability to study environments in which coalition members jointly produce and share output: we characterize sharing rules that induce stability and we derive qualitative properties of the resulting stable coalition structures. The condition for stability allows for complementarities and peer effects, and it makes their analysis tractable. In particular, we check how the presence of complementarities and peer effect affects the properties of stable matchings.

We focus on environments in which agents care only about the composition of their coalition and are not affected by other coalitions that might form. In these environments, the set of stable coalition structures is determined by the profile of agents' preferences over coalitions. We can thus identify an environment with the set of agents' preference profiles that are possible in that environment. The main results presume that the environment contains a rich variety of possible preference profiles so that, roughly speaking, any agent can rank the coalitions he or she belongs to in all possible ways. For instance, consider an environment in which agents first learn what outputs coalitions can produce, and then form coalitions, produce output, and share the output equally. In this environment, the agents' preferences over coalitions depend on the profile of coalitional outputs, and the resulting set of preference profiles is rich if all nonnegative profiles of coalitional outputs are possible. This example is developed in Section 2. We, furthermore, focus on matching environments in which each firm has a capacity to hire at least two workers and on one-sided coalition formation in which every three agents can form a coalition.[3]

The main results—Theorems 1 and 2—establish that there is a stable coalition structure for all preference profiles if and only if agents' preferences are pairwise aligned in all preference profiles.[4] Agents' preferences are pairwise aligned if any two agents rank proper coalitions that contain both of them in the same way. For instance, a firm prefers a firm-and-one-worker coalition to a larger coalition if and only if the worker does. We further show that if pairwise alignment is satisfied and preferences are strict, then the stable coalition structure is unique in many-to-one matching, and there are at most two stable coalition structures in one-sided coalition formation.

We apply these main results to characterize sharing rules that induce stability in matching and coalition formation environments in which coalition members

[3]For a firm to perceive two workers as complementary—or for the workers to experience peer effects—the firm must be able to employ at least two workers. Similarly, for an agent to see others as complementary in one-sided coalition formation, the coalition must be at least of size 3. Hence, our assumptions are without much loss of generality for the analysis of complementarities and peer effects. The assumptions also simplify the formulation of our other results, and the two main problems we are excluding—one-to-one matching and the roommate problem—are both fairly well understood.

[4]Both directions of implication rely on the richness assumption, which plays a similar role to the uniform-domain assumptions of Arrow (1951) and Nash (1950).

jointly produce and share output. As in the example above, agents' payoffs in each coalition are determined by the coalitional output and a sharing rule. A sharing rule is a collection of functions—one for each coalition–member pair—that map coalitional output to the member's share of the output. When the sharing rule is fixed, each profile of outputs determines a profile of agents' preferences over coalitions. The domain of preference profiles obtained as we vary outputs is rich provided that the functions defining the sharing rule are continuous and monotonic, and that each member's share can be arbitrarily large when the coalition's output is sufficiently large. Hence, the main stability results imply that such a sharing rule leads to stable coalition formation problems if and only if it generates pairwise-aligned preference profiles.[5]

Some sharing rules generate pairwise-aligned profiles, but others do not. For instance, preferences are pairwise aligned if agents' shares in each coalition are determined through equal sharing, multiagent Nash bargaining, Tullock's (1980) rent-seeking game, or egalitarian and Rawlsian sharing rules, but not if the shares are determined through Kalai–Smorodinsky bargaining. Our stability results thus imply that if the shares are determined through, say, Nash bargaining, then there always exists a stable coalition structure. They also imply that if the shares are determined through Kalai–Smorodinsky bargaining, then there is no stable coalition structure for some output profiles.

The above existence results build on earlier work by Farrell and Scotchmer (1988), who established the existence of stable coalition structures for equal sharing, and by Banerjee, Konishi, and Sönmez (2001), who established the existence of stable structures for some other linear sharing rules.[6] We go beyond their results by establishing the general and tight connection between pairwise-aligned rules and stable outcomes, and thus, for instance, showing that Nash bargaining implies stability.

[5]Among efficient and monotonic sharing rules on proper coalitions, rules that generate pairwise-aligned preferences are precisely those which are consistent in the sense of Harsanyi (1959) and Thomson and Lensberg (1989). We thus establish a link between stability and consistency. Prior work on consistency and the core is only superficially related: studies such as Peleg (1986) examined consistency properties of the core itself, assuming the core is nonempty (cf. Thomson's (2009) survey), while our results imply that the consistency of the underlying sharing rule is linked to the nonemptiness of the resultant core.

[6]Farrell and Scotchmer observed that, under equal sharing, the per-agent payoff from a coalition is the same for all agents, and thus there is a common ranking of all coalitions that agrees with agents' rankings. They then concluded that there exists a stable coalition structure because the coalition at the top of the common ranking cannot be blocked, and then the top coalition among the remaining agents cannot be blocked, and so on. When specialized to the sharing rules setting, our existence results go beyond Farrell and Scotchmer's by replacing equal sharing with pairwise alignment and by showing that in this way we obtain the entire class of stability-inducing sharing rules. A (relaxed) analog of their common ranking structure is true in our setting; the construction of such a relaxed common ranking is central to our uniqueness argument. Section 5.2 compares our results on assortativeness with those of Farrell and Scotchmer. None of our other results has a counterpart in Farrell and Scotchmer (1988) or in prior developments of their setting.

We then study qualitative properties of stable coalition structures formed when coalition members share output according to stability-inducing sharing rules. As an auxiliary result, we show that each pairwise-aligned rule that efficiently shares the output in each coalition can be represented by a profile of agents' "bargaining functions." As in Nash bargaining, members of each proper coalition share output as if they were maximizing the product of their bargaining functions.[7]

The coalition structures formed under stability-inducing sharing rules are assortative: agents sort themselves into coalitions according to their productivity and the Aumann and Kurz (1977) fear-of-ruin coefficient of their bargaining functions.[8] Agents with similar bargaining functions tend to belong to the same coalitions in the stable structure.[9] Thus, roughly speaking, we should observe less variation in bargaining power within a coalition than within the society at large.

Because the sharing rules are exogenous, agents face a holdup problem: some beneficial coalitions may not form because agents with strong bargaining power are not able to commit to reward adequately agents with relatively weak bargaining power. We discuss holdup in the concluding remarks.

Our last result on sharing rules compares stable coalitions across different stability-inducing sharing rules. Assuming that outputs are independently drawn from distributions with monotone hazard rates and that there are no ex ante productivity differences among coalition members, this paper shows that the probability of a coalition being stable is larger when the bargaining functions of coalition members are more equal.[10] This result provides a first step toward understanding how inequality of bargaining functions affects the distribution of coalition sizes.

[7]Translated in terms of consistency (see footnote 5), the above characterization extends the main insight of Lensberg (1987) to many-to-one matching.

[8]When we control for productivity, the results imply that in many-to-one matching and in one-sided coalition formation, stability itself is a sufficient condition for monotonic sorting on bargaining functions. This contrasts with sorting on exogenous productivity types, where Becker (1973) and the subsequent literature established supermodularity-type conditions that ensure that agents sort monotonically in their types (see, for instance, Shimer and Smith (2000), Legros and Newman (2007), and Eeckhout and Kircher (2010)).

[9]In particular, if agents share output in Nash bargaining and are endowed with constant relative risk aversion utilities, then they match in a positive assortative way according to risk aversion. This remains true in one-to-one matching, and is of interest in the context of the large empirical literature on risk-sharing and its recent critique by Chiappori and Reny (2006). Observing that the literature relies on the implicit assumption that agents match in a positive assortative way according to risk aversion, Chiappori and Reny argued that this is impossible in a general model of one-to-one matching. Their result turns out to hinge on the assumption—shared by the empirical literature they criticize, but not by this paper—that agents can freely contract on the sharing of output.

[10]The assumption of symmetry of the distribution of coalition members' productivity is important. See Abramitzky (2008) for an empirical analysis of how productivity differences among members affect the stability of Israeli equal-sharing kibbutzim.

The rest of the paper applies its main results to analyze many-to-one matching with complementarities and peer effects. In our framework, firms see workers as complementary when the complementarity is embedded in the profile of outputs. Workers then care who their peers are. The framework can thus be used to model a newspaper hiring workers for a reporting assignment: if both a reporter and a photographer are available, the newspaper wants to hire the two of them over a generalist who can both write the article and take the accompanying pictures reasonably well; if, however, no photographer is available, the newspaper might prefer to withhold the offer from the reporter and hire the generalist instead. In doing so, the newspaper treats the reporter and the photographer as complementary. Similarly, the framework can be used to model a physician who wants to open a practice if there are enough patients who would choose him or her as a primary care provider; if there too few, however, the physician might choose not to treat any of them. By allowing peer effects, we allow workers to care about interactions in the workplace and allow the worker's workload to be influenced by their peers. In the model of matching between physicians and patients, we allow patients to care about the congestion in their physician's office. Reinterpreted in terms of schools and students whose preferences are determined by expected schooling outcomes, our framework allows schooling outcomes to be affected by peer effects.[11]

By allowing complementarities and peer effects, we depart from the standard treatment of many-to-one matching. In the matching literature it is standard to assume that firms perceive workers as gross substitutes—if a firm chooses to employ a worker w from a pool of workers, then the firm chooses to employ w from any subpool containing w—and that workers (or students or doctors) care only about the identity of the firm with which they match, but not about the identity of their peers. The gross-substitutes and absence-of-peer-effects assumptions have been developed by Gale and Shapley (1962), Kelso and Crawford (1982), Roth (1985), and Hatfield and Milgrom (2005), among others.

We show that allowing complementarities and peer effects changes the standard comparative statics of many-to-one matching. For instance, in the standard model, retiring an agent from one side of the market benefits other agents on the same side and hurts agents on the opposite side of the market (Crawford (1962)). We show that this result is no longer true in the presence of complementarities; in the example above, the reporter might be hurt when the photographer retires.

Finally, we study implementation of stable matchings and coalition structures. We show that as long as the pairwise-alignment condition is satisfied, and irrespective of whether complementarities are present, strategic agents reach a stable matching or coalition structure on the equilibrium path of a wide variety

[11]Whether peer effects affect schooling outcomes is, of course, an empirical question; see, for instance, Case and Katz (1991), Sacerdote (2001), and Angrist and Lang (2004).

of non-cooperative games such as Gale and Shapley's deferred acceptance algorithm. This positive finding contrasts with results from the standard model, in which strategic agents may reach an unstable outcome (Dubins and Freedman (1981), Roth (1982), and Ma (2010)).

The present paper is the first to propose the pairwise-alignment assumption and analyze the impact of complementarities on comparative statics and strategic play. Prior results on complementarities in matching focused on the impossibility of obtaining existence when complementarities are allowed. For instance, in a very general model of matching with contracts, Hatfield and Milgrom (2005) and Hatfield and Kojima (2008) showed that a variant of the gross substitutes condition is the most general condition that—when imposed separately on the preferences of each agent—guarantees the existence of a stable matching.[12] Our results are consistent with their conclusion because the pairwise-alignment assumption is a constraint on the relation between preferences of pairs of agents rather than on the preferences of individual agents. For the study of complementarities, it is likely necessary to consider relations between preferences and how they are co-determined by the matching environment.

There are few positive results that go beyond the standard assumptions of gross substitutes and absence of peer effects. Echenique and Oviedo (2004) and Echenique and Yenmez (2007) constructed algorithms that find stable matchings whenever they exist. Dutta and Massó (1997) allowed some peer effects but maintained substitutability.[13] Kojima, Pathak, and Roth (2010) showed that the peer effects created by the presence of married couples become less problematic as the market becomes large.

2. EXAMPLE

Before turning to the more general model, let us examine the questions and results of the paper in the context of a simple matching environment with four agents and two states of nature. Looking at three illustrative sharing rules will give us a preliminary sense of which rules give us stable coalition structures and which do not. We will also see how the stable coalition structures depend on the sharing rule.

[12]Hatfield and Milgrom, and Hatfield and Kojima assumed that there are no peer effects. Klaus and Klijn (2005), and Sönmez and Unver (2010) proved related impossibility results. Ma (2001) provided an example of empty core when the substitutes condition is satisfied but peer effects are present.

[13]Dutta and Massó (1997) weakened the absence-of-peer-effects condition in two separate ways: (i) allowing exogenously "married" worker couples to prefer any coalition that includes their partner over any other coalition, and (ii) allowing peer effects to influence workers' preferences between two coalitions involving the same employer (firm) but not otherwise. Revilla (2007) further developed this line of research.

TABLE I

COALITIONAL OUTPUTS IN STATES ω_1 AND ω_2

	$\{f, 1, 2\}$	$\{g, 1, 2\}$	$\{f, 1\}$	$\{f, 2\}$	$\{g, 1\}$	$\{g, 2\}$	$\{f\}$	$\{g\}$	$\{1\}$	$\{2\}$
State ω_1	72	42	44	64	1	1	0	0	0	0
State ω_2	42	99	44	64	1	1	0	1	0	0

In our example, there are two firms f and g, and two workers 1 and 2. They are forming coalitions so as to produce output. Each firm can employ either one or both workers. Agent a derives utility (or profit) $U_a(s)$ from obtaining share s of output. Let us assume that

$$U_f(s) = s^{1/2}, \quad U_g(s) = s, \quad U_1(s) = s^{1/6}, \quad U_2(s) = s^{1/2}.$$

We consider two states of nature ω_1 and ω_2. Agents know the state when they form coalitions; in particular they know the outputs that each coalition can produce if formed. The outputs are given in Table I. Thus, for instance, we assume that coalition $\{f, 1, 2\}$ produces 72 in state ω_1 and produces 42 in state ω_2. We denote by $\mathbf{y}(C, \omega)$ the output coalition C produces in state ω.

The output in each coalition in the resulting many-to-one matching is divided according to a sharing rule. One interpretation of the sharing rule setup is that the coalition formation and production take place on two different dates. On date 1, the agents learn the state of nature $\omega \in \{\omega_1, \omega_2\}$ and then form coalitions. On this date, they cannot make transfers conditional on joining a coalition and cannot affect the payoffs they will obtain on date 2. In effect, on date 1, the agents' preferences over coalitions reflect their shares of the output produced on date 2.

Let us look at the following three sharing rules.

Equal Sharing

Agents share output equally: in state of nature ω, the share of agent i in coalition C is $\frac{\mathbf{y}(C,\omega)}{|C|}$.

Agents' shares of output in the four productive coalitions (rounded to the first decimal point) are listed in Table II. Under equal sharing, there is a unique stable matching in state ω_1 and it takes the form $\{\{f, 2\}, \{g, 1\}\}$. In state ω_2, the unique stable matching is $\{\{f\}, \{g, 1, 2\}\}$. As first observed by Farrell and Scotchmer (1988) in their study of partnerships that share profits equally, it is not a coincidence that there is a stable matching in both states of nature when agents share output equally. Since each agent wants to be in a coalition with output per agent, $(\mathbf{y}(C, \omega))/|C|$, as high as possible, no agent wants to change a coalition with the highest output per agent. We can hence set this coalition aside and recursively construct a stable coalition structure.

TABLE II

OUTPUT SHARES AND STABLE MATCHINGS FOR DIFFERENT SHARING RULES AND STATES[a]

	$\{f, 1, 2\}$	$\{g, 1, 2\}$	$\{f, 1\}$	$\{f, 2\}$	Stable Matching
State ω_1					
Equal sharing	24, **24**, 24	**14**, 14, 14	22, 22	**32**, **32**	$\{f, 2\}$, $\{g, 1\}$
Nash bargaining	31, 10, **31**	**25**, 4, 13	**33**, **11**	32, 32	$\{f, 1\}$, $\{g, 2\}$
Kalai–Smorodinsky	**33**, 7, **33**	**25**, 2, 15	30, **14**	32, 32	$\{f, 1, 2\}$, $\{g\}$
State ω_2					
Equal sharing	14, 14, 14	**33**, **33**, **33**	22, 22	**32**, 32	$\{f\}$, $\{g, 1, 2\}$
Nash bargaining	18, 6, 18	**59**, 10, 30	**33**, **11**	32, **32**	$\{f, 1\}$, $\{g\}$, $\{2\}$
Kalai–Smorodinsky	19, 4, 19	**59**, 5, **35**	30, **14**	**32**, 32	Does not exist

[a] Agents' shares in coalitions are rounded and listed in the same order as agents. Each agent's highest share is in boldface type.

Nash Bargaining

Agents share output according to multiagent Nash bargaining: in state of nature ω, members of coalition C obtain shares s_a that maximize

$$\max_{s_a \geq 0} \prod_{a \in C} U_a(s_a)$$

subject to

$$\sum_{a \in C} s_a \leq \mathbf{y}(C, \omega).$$

We may interpret the function U_a as the composition of agent a's bargaining power and utility (or profit) function.[14]

In both states of nature there is a unique stable matching: $\{\{f, 1\}, \{g, 2\}\}$ in state ω_1 and $\{\{f, 1\}, \{g\}, \{2\}\}$ in state ω_2. It turns out that, again, it is not a coincidence that Nash bargaining leads to stable matchings in both states of nature. For any profile of outputs, we can construct a stable matching. The construction does not depend on the special power-function form of the utility functions U_a. Let us take any increasing, concave, and differentiable utility functions, normalize them so that $U_a(0) = 0$, and proceed in three steps. First, observe that the so-called fear-of-ruin coefficient (Aumann and Kurz (1977)) $\chi_a(s_a) = U_a(s_a)/U_a'(s_a)$ is the same for every agent a in any given coalition C (because the first order condition in the Nash bargaining maximization equalizes $1/\chi_a$ and the Lagrange multiplier) and denote by χ_C this common fear of

[14] We assume that agents' outside options are 0. This is not important. Nash bargaining leads to stable matchings whenever the outside options are exogenous or only depend on the outputs in single-agent coalitions.

ruin. Second, observe that each agent's allocation s_i is increasing in the common fear of ruin χ_C of agents in coalition C. Third, conclude that no agent wants to change a coalition that maximizes χ_C and, therefore, we can set a coalition with maximal χ_C aside and look at coalition formation among the remaining agents. In this way, we can recursively construct a stable coalition structure.

Do all sharing rules lead to stable outcomes? The answer is "No." Consider a third illustrative sharing rule.

Kalai–Smorodinsky Bargaining

Agents share output according to Kalai–Smorodinsky bargaining: in state of nature ω, members of coalition C obtain shares s_a such that

$$\frac{U_a(s_a)}{U_a(\mathbf{y}(C,\omega))} \quad \text{is constant across } a \in C \text{ and } \sum_{a\in C} s_a = \mathbf{y}(C,\omega).$$

Under Kalai–Smorodinsky bargaining, there is a unique stable matching $\{\{f,1,2\},\{g\}\}$ in state ω_1. In state ω_2, however, there is no stable matching. Indeed, any stable matching would need to include one of the coalitions $\{f,1\}, \{f,2\}, \{g,1,2\}$ because $\{f,1,2\}$ is dominated by both $\{f,1\}$ and $\{f,2\}$, and the remaining coalitions produce small or zero outputs. However, none of the coalitions $\{f,1\}, \{f,2\}, \{g,1,2\}$ can be part of a stable matching for the following reasons:

- Coalition $\{g,1,2\}$ would be blocked by worker 1 and firm f.
- Coalition $\{f,1\}$ would be blocked by firm f and worker 2.
- Coalition $\{f,2\}$ would be blocked by worker 2 together with firm g and worker 1.

The above three sharing rules illustrate the results of the paper. First, some sharing rules consistently lead to stable outcomes and others do not. The pairwise alignment of preferences turns out to be the key differentiating factor between equal sharing and Nash bargaining, which lead to stable outcomes irrespective of the profile of coalitional outputs, and the Kalai–Smorodinsky bargaining, which does not. Preferences are pairwise aligned for both equal sharing and Nash bargaining, irrespective of the profile of outputs. The pairwise alignment fails, however, for Kalai–Smorodinsky bargaining in state ω_1: firm f prefers $\{f,1,2\}$ over $\{f,1\}$, while worker 1 has the opposite preference. This is not a coincidence: our first insight is that the pairwise-aligned and the stability-inducing rules coincide. Section 4 establishes this result for the more general setting of domains of ordinary preference profiles.

Second, we show that Nash bargaining is a typical example of stability-inducing sharing rules. Each of them may be described by endowing agents with a profile of increasing, differentiable, and log-concave bargaining functions, and letting them share the output so as to maximize the analog of the Nash product (see Section 5.1).

Third, notice that under Nash bargaining, worker 2 is weaker than worker 1 and firm f is weaker than firm g. In both states, the weak worker matches with the weak firm, and the strong worker either matches with the strong firm or remains unmatched. Section 5.2 shows that such an assortative structure is typical for stability-inducing sharing rules.

Fourth, in state ω_2, workers 1 and 2 are in the same coalition under equal sharing but not under Nash bargaining. Again, this is illustrative of a typical situation. Section 5.3 shows that the more equal is the sharing among a group of workers, the more likely they are to be in the same coalition in the stable matching or coalition structure.

Finally, notice that workers 1 and 2 are complementary for firm g in state ω_2 under both equal sharing and Nash bargaining, and that each of the workers cares whether the other is also hired by the same firm. Hence, the setting encompasses environments in which there are complementarities and peer effects. We examine matching with complementarities and peer effects in Section 6.

3. MODEL

Let A be a finite set of agents and let $\mathcal{C} \subseteq 2^A$ be a set of coalitions. A coalition C is proper if $C \neq A$. Each agent $a \in A$ has a preference relation $\precsim_a$ over coalitions C that contain a. The profile of preferences of agents in A is denoted $\precsim_A = (\precsim_a)_{a \in A}$. All references to a coalition in this paper presume that the coalition belongs to $\mathcal{C}$ and all preference comparisons $C \precsim_a C'$ presume that $a \in C \cap C'$.

A *coalition structure* μ is a partition of A into coalitions from $\mathcal{C}$. We assume throughout that there exists at least one coalition structure. This assumption is satisfied if, for instance, every singleton set is a coalition. A coalition structure μ is *blocked* by a coalition C if each agent $a \in C$ strictly prefers C to the coalition in μ that contains a. A coalition structure is *stable* if no coalition blocks it. This is the standard notion of core stability from the matching literature.

Let $\mathbf{R}$ be a subset of the Cartesian product of sets of preference profiles of agents in A; we call $\mathbf{R}$ a preference domain. We do not require $\mathbf{R}$ to be Cartesian.

Our main existence results generalize and formalize the comparison of sharing rules from Section 2, and taken together say that—with enough coalitions in $\mathcal{C}$ and preference profiles in $\mathbf{R}$—all preferences in the domain admit stable coalition structures if and only if (iff) the preferences are pairwise aligned. Preferences are *pairwise aligned* if for all agents $a, b \in A$ and proper coalitions C, C' that contain a, b, we have

$$C \precsim_a C' \iff C \precsim_b C'.$$

Preferences are *pairwise aligned over the grand coalition* if either $A \notin \mathcal{C}$, or $A \in \mathcal{C}$ and the above equivalence is true whenever C or C' equals A. Notice that the pairwise alignment implies that $C \sim_a C'$ iff $C \sim_b C'$ and $C \succ_a C'$

iff $C \succ_b C'$. The pairwise alignment of preferences when agents share output in Nash bargaining was noted already by Harsanyi (1959), and it is also straightforward for equal sharing. The pairwise alignment of preferences when agents share output in Kalai–Smorodinsky bargaining fails in state ω_1 of Section 2.

3.1. *Regular Families of Coalitions*

The family of coalitions $\mathcal{C}$ is *regular* if there is a partition of the set of agents A into two disjoint, possibly empty, subsets F (firms) and W (workers) that satisfy the following three assumptions:

C1. For any two different agents, there exists a coalition containing them if and only if at least one of the agents is a worker.

C2. For any workers a_1, a_2 and agent a_3, there exist proper coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ such that $C_{k,k+1} \ni a_k, a_{k+1}$ and $C_{1,2} \cap C_{2,3} \cap C_{3,1} \neq \emptyset$.

C3. (i) For any worker w and agent a, if $\{a, w\}$ is not a coalition, then there are two different firms f_1, f_2 such that $\{f_1, a, w\}$ and $\{f_2, a, w\}$ are coalitions. (ii) No proper coalition contains W.

Assumption C1 imposes a many-to-one structure that is more general than the standard many-to-one matching in that it does not assume away coalitions composed entirely of workers. Assumptions C2 and C3(i) guarantee that either workers can form three-worker coalitions or firms can hire at least two workers.[15] The separation of C2 and C3(i) and the partial overlap between C1 and C3(i) allow precise matching between assumptions and results: some of the results rely on only one or two of the assumptions. Assumption C3(ii) guarantees that no firm can hire all workers.

Assumptions C1–C3 are satisfied in many coalition formation environments. In particular, the assumptions are motivated by the following two standard environments:

- The unconstrained one-sided coalition formation defined by $\mathcal{C} = 2^A - \{\emptyset\}$.
- Many-to-one matching defined as follows: the set of agents is partitioned into two subsets F (interpreted as the set of firms, colleges, or hospitals) and W (interpreted as the set of workers, students, or doctors), each agent $f \in F$ is endowed with a capacity constraint M_f, and $\mathcal{C} = \{\{f\} \cup S : f \in F, S \subseteq W, |S| \leq M_f\} \cup \{\{w\} : w \in W\}$.

The unconstrained coalition formation satisfies C1–C3 with $W = A$, $F = \emptyset$. We prove C2 by setting $C_{1,2} = C_{2,3} = C_{3,1} = \{a_1, a_2, a_3\}$. The remaining assumptions are straightforward. Many-to-one matching satisfies the assumptions as long as $M_f \in \{2, \ldots, |W| - 1\}$ and $|F| \geq 2$. Indeed, C1 is straightforward. Assumption C2 requires $M_f \geq 2$ for all $f \in F$, and is established by setting

[15]In analysis of complementarities and peer effects, assumptions C2 and C3(i) are without much loss of generality (see footnote 3). Assumption C2 is implied by the following simpler requirement: for any agents $a_1, a_2 \in W$, $a_3 \in A$, there exists a proper coalition containing them.

$C_{1,2} = C_{2,3} = C_{3,1} = \{a_1, a_2, a_3\}$ if $a_3 \in F$, and $C_{k,k+1} = \{a_k, a_{k+1}, f\}$ for some $f \in F$ if $a_3 \in W$. Assumption C3(i) requires $M_f \geq 2$ for at least two $f_1, f_2 \in F$, and is true as $\{a, w\}$ is a coalition if $a \in F$, and $\{f_1, a, w\}$ and $\{f_2, a, w\}$ are coalitions if $a \in W$. Assumption C3(ii) requires $M_f < |W|$ for all $f \in F$.[16]

Assumptions C1 and C2 are used to show that pairwise alignment is sufficient for stability, and assumptions C1 and C3 are used to show that pairwise alignment is necessary for stability. The equivalence between stability and pairwise alignment requires some assumptions on the family of coalitions, as illustrated by the following example of the roommate problem.

EXAMPLE 1: The roommate problem is the coalition formation problem in which $\mathcal{C} = \{C \subseteq A, |C| \leq 2\}$. Any preference profile in the roommate problem is pairwise aligned, but the existence of a stable coalition structure is not assured. For instance, there is no stable coalition structure if $A = \{a_1, a_2, a_3\}$, all agents prefer any two-agent coalition to being alone, and their preferences among two-agent coalitions are such that

$$\{a_1, a_2\} \succ_{a_2} \{a_2, a_3\} \succ_{a_3} \{a_3, a_1\} \succ_{a_1} \{a_1, a_2\}.$$

3.2. *Rich Domains of Preference Profiles*

A domain of preference profiles $\mathbf{R}$ is called *rich* if it satisfies the following assumptions:

R1. For any profile $\precsim_A \in \mathbf{R}$, any agent a, and any three different coalitions C_0, C, C_1, if $C_0 \precsim_a C_1$ and $a \in C$, then there is a profile $\precsim'_A \in \mathbf{R}$ such that $C_0 \precsim'_a C \precsim'_a C_1$ and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

R2. (i) For any $\precsim_A \in \mathbf{R}$ and two different coalitions C, C_1, there is a profile $\precsim'_A \in \mathbf{R}$ such that $C \prec'_a C_1$ for all $a \in C \cap C_1$ and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

(ii) For any $\precsim_A \in \mathbf{R}$, any agents a, b, and any three different coalitions C_0, C, C_1, if $C_0 \prec_a C \sim_b C_1$, then there is a profile $\precsim'_A \in \mathbf{R}$ such that $C_0 \prec'_a C \prec'_b C_1$ and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

The assumptions R1 and R2 formalize the requirement that there is substantial variability, or richness, among preference profiles. Assumption R1 requires that for any preference profile in $\mathbf{R}$ and any coalition C, there is a (shocked)

[16]The marriage problem is a well known special case of many-to-one matching defined by $M_f = 1$ for all $f \in F$. While the marriage problem does not satisfy assumptions C2 and C3, we exclude it primarily for the sake of simplicity of exposition: the equivalence between stability and the pairwise alignment obtains for the marriage problem. The pairwise alignment of preferences is satisfied for the marriage problem in a trivial way, and Gale and Shapley (1962) showed that the marriage problem always admits a stable matching (coalition structure). On the other end of the matching literature, many-to-many matching, in general, is not a coalition formation problem.

preference profile in **R** in which C is ranked just below another coalition C_1 by the relevant agent. Assumption R2(i) postulates the existence of a shocked profile in which a coalition C is ranked below another coalition C_1 by all relevant agents. Assumption R2(ii) postulates that there is a shocked profile in which an indifference is broken; this last assumption is trivially satisfied if agents' preferences are always strict.

Examples of rich preference domains include the domain of preference profiles generated in equal sharing and Nash bargaining when we vary output functions $\mathbf{y}:\mathcal{C} \to (0, \infty)$. The domain of profiles generated in Kalai–Smorodinsky bargaining is rich provided each agent's utility is unbounded above or provided each agent's utility is bounded above; the Kalai–Smorodinsky domain might fail condition R1 if some agents' utilities are unbounded, while others are bounded. More generally, consider a setting in which agents' payoffs are determined by a state of nature $\omega \in \times_{C\in\mathcal{C}} \Omega_C$ and the payoffs in coalition C depend only on the C-coordinate of the state of nature. For each coalition C and agent $a \in C$, consider a payoff mapping $D^{\text{payoff}}_{a,C}$ from Ω_C to payoffs of agent a. Then R1 is satisfied as long as the set of outcomes $D^{\text{payoff}}_{a,C}(\Omega_C)$ does not depend on coalition C, but only on agent a. Assumption R2 is satisfied if we additionally require that for each $C \in \mathcal{C}$, the set Ω_C is an open interval in R, and the payoff mapping is continuous and strictly monotonic. Other examples satisfying both R1 and R2 include the domain of all strict preference profiles and the domain of all preference profiles.

Assumption R1 is used to show that pairwise alignment is sufficient for stability, and assumption R2 is added to show that pairwise alignment is necessary for stability. To see that the relationship between stability and pairwise alignment needs some assumptions on the domain of preferences, recall Kalai–Smorodinsky bargaining from Section 2. In state ω_2, there is no stable matching even though, in this state, the pairwise alignment holds. At the same time, in state ω_1 the pairwise alignment fails even though, in this state, there is a stable matching. The next section and the Supplemental Material (Pycia (2012)) provide more details on the role of assumptions C1–C3, R1, and R2.

4. MAIN RESULTS: STABILITY IN PREFERENCE DOMAINS

Our main existence results are the following theorems.

THEOREM 1: *Suppose that the family of coalitions $\mathcal{C}$ satisfies* C1 *and* C2, *and that the preference domain* **R** *satisfies* R1. *If all preference profiles in* **R** *are pairwise aligned*, *then* (i) *all $\precsim_A \in$* **R** *admit a stable coalition structure and* (ii) *the stable coalition structure is unique for any profile of strict preferences $\precsim_A \in$* **R** *that is pairwise aligned over the grand coalition*.

THEOREM 2: *Suppose that the family of coalitions $\mathcal{C}$ satisfies* C1 *and* C3, *and that the preference domain* **R** *satisfies* R1 *and* R2. *If all profiles from* **R** *admit stable coalition structures*, *then all profiles from* **R** *are pairwise aligned*.

Theorem 1 relies on part R1 of the richness assumption. To develop an understanding of the role of this assumption, let us look again at the failure of stability under the Kalai–Smorodinsky sharing rule from Section 2. Theorem 1 implies that the preference profile $\precsim_{\{f,g,1,2\}}$ of agents using the Kalai–Smorodinsky rule to share outputs in state ω_2 cannot be embedded in an R1-rich domain of pairwise-aligned preference profiles. To check this corollary, notice that if this profile belonged to an R1-rich domain of pairwise-aligned profiles, then there would exist a pairwise-aligned profile $\precsim'_{\{f,g,1,2\}}$ such that

$$\{f,1\} \precsim'_f \{f,1,2\} \precsim'_f \{f,2\},$$

and all agents' $\precsim'_{\{f,g,1,2\}}$-preferences between pairs of coalitions not including $C = \{f,1,2\}$ would be the same as their $\precsim_{\{f,g,1,2\}}$-preferences. Then the pairwise alignment of $\precsim'_{\{f,2\}}$ would imply that $\{f,1,2\} \precsim'_2 \{f,2\} \prec'_2 \{g,1,2\}$, and the transitivity of $\precsim'_2$ and the pairwise alignment of $\precsim'_{\{1,2\}}$ would give

$$\{f,1,2\} \prec'_1 \{g,1,2\} \prec'_1 \{f,1\}.$$

However, then the $\precsim'_{\{f,1\}}$-preferences of agents 1 and f between coalitions $\{f,1,2\}$ and $\{f,1\}$ would violate pairwise alignment. This contradiction shows that $\precsim_{\{f,g,1,2\}}$ cannot be embedded in any R1-rich domain of pairwise-aligned preferences.

The first step of the proof of Theorem 1 generalizes the above argument to show that the pairwise alignment, R1, C1, and C2 imply that there are no 3-cycles for any $\precsim_A \in \mathbf{R}$. A 3-cycle or, generally, an n-cycle is any configuration of proper coalitions $C_1, \dots, C_n$ and agents $a_1, \dots, a_n$ such that (subscripts modulo n)

$$(1) \qquad C_i \precsim_{a_i} C_{i+1} \quad \text{for} \quad i = 1, \dots, n, \text{ with at least one preference strict.}$$

For instance, the Section 2 discussion of the Kalai–Smorodinsky rule shows that in state ω_2, coalitions $\{g,1,2\}, \{f,1\}, \{f,2\}$ form a 3-cycle.

The main step of the proof uses the lack of 3-cycles, R1, and C1 to show that lack of n-cycles implies lack of $(n+1)$-cycles, and hence that there are no n-cycles for $n = 2, 3, \dots$. The final step of the proof is to observe that the lack of n-cycles implies both the existence and—with the added assumptions of strict preferences and pairwise alignment over the grand coalition—the uniqueness of stable coalition structure. The uniqueness relies on the added assumptions. For instance, if $|A| \geq 3$ and $\mathcal{C} = \{C \subseteq A, C \neq \emptyset\}$, then there is a domain of pairwise-aligned profiles that contains a strict preference profile that (i) is not pairwise aligned over the grand coalition and (ii) allows both the grand coalition A and a coalition structure of proper coalitions to be stable.

The proof of the necessity of pairwise alignment (Theorem 2) roughly reverses the steps of the proof of its sufficiency. First, assuming R2, we show that stability implies lack of 3-cycles C_1, C_2, C_3 such that $C_j \cap C_i$ are singletons for $j \neq i$. Then, assuming C1, C3, and R1, we show that the lack of such 3-cycles implies pairwise alignment. To get a sense of this proof, consider the preference profile that obtains in state ω_1 when agents share outputs in Kalai–Smorodinsky bargaining and assume that the environment contains a third worker, 3. Since agents f and 1 differ in their preference ranking of coalitions $\{f, 1\}$ and $\{f, 1, 2\}$, Theorem 2 implies that this preference profile cannot be embedded in a rich domain of stability-inducing profiles. Let us check this claim directly under an additional assumption that all preference profiles are strict. Any domain satisfying R1 and R2(ii), and containing the ω_1-profile contains a preference profile $\precsim_A$ such that $\{f, 1\} \prec_f \{f, 3\} \prec_f \{f, 1, 2\}$ and $\{f, 1, 2\} \prec_1 \{g, 1, 3\} \prec_1 \{f, 1\}$. Consider the case $\{f, 3\} \prec_3 \{g, 1, 3\}$; the other case is symmetric. In this case, $\{f, 3\} \prec_3 \{g, 1, 3\} \prec_1 \{f, 1\} \prec_f \{f, 3\}$ is a 3-cycle. Because of R2(i), we may assume that members of coalitions $\{f, 3\}$, $\{g, 1, 3\}$, and $\{f, 1\}$ strictly prefer them to any coalition other than these three. Any stable coalition structure would then need to contain one of these three coalitions, but each one of them is blocked by one of the other two. Hence, there is no stable coalition structure.

The Appendix gives the proofs and the Supplemental Material discusses the trade-offs involved in relaxation of the assumptions. For instance, regularity assumption C2 may be dropped in Theorem 1 at the cost of replacing pairwise alignment with lack of 3-cycles. In some problems, such as many-to-one matching, the richness assumptions may be relaxed to take account of the additional structure of such problems. The results hold true for other stability concepts such as pairwise stability and group stability in many-to-one matching. Finally, the above map of the proofs implies that no preference profile in a rich domain of pairwise-aligned profiles admits an n-cycle. The Supplemental Material also demonstrates that every profile that does not admit n-cycles may be embedded in a rich domain of pairwise-aligned profiles.

5. APPLICATIONS: SHARING RULES

This section applies the existence results of the previous section to an analysis of stability-inducing sharing rules. As in Section 2, we look at instances of our general setup in which each coalition produces output $\mathbf{y}(C) \in R_+ = [0, +\infty)$. The mapping from coalitional outputs to agents' preferences is determined by a sharing rule. A *sharing rule* is a collection of functions $D_{a,C} : R_+ \to R_+$, one for each coalition C and each of its members $a \in C$, that map the output of C into the share of output obtained by agent a. We denote the sharing rule given by functions $D_{a,C}$ as $D = (D_{a,C})_{C \in \mathcal{C}, a \in C}$. We assume that the shares are feasible, $\sum_{a \in C} D_{a,C}(y) \leq y$.[17] A sharing rule is *pairwise aligned* if the pref-

[17]Embedded in this assumption is the idea that transfers between members of a coalition are costless. Corollary 1 remains true when the transfers are costly.

erence profiles that it generates are pairwise aligned for every profile of outputs. In Section 2 we have seen two examples of pairwise-aligned sharing rules (equal sharing and Nash bargaining) and one instance of a sharing rule that violates pairwise alignment (Kalai–Smorodinsky).

Theorems 1 and 2 imply the following corollary.

COROLLARY 1: *Suppose that the family of coalitions $\mathcal{C}$ satisfies* C1–C3 *and the functions $D_{a,C}$ are strictly increasing and continuous*, *and* $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$ *for all $C \in \mathcal{C}$, $a \in C$. Then there is a stable coalition structure for each profile of outputs if and only if the sharing rule D is pairwise aligned*.

For instance, in the environment of Section 2, the assumptions of the corollary are satisfied by all three sharing rules. To prove the corollary, first notice that the pairwise alignment yields stability because the domain of preference profiles generated by D satisfies R1 and hence Theorem 1 is applicable. Second, to prove the converse implication, notice that the restriction of the sharing rule D to profiles of strictly positive outputs satisfies R1 and R2, and hence Theorem 2 implies that agents' preferences over coalitions with strictly positive outputs are pairwise aligned. This implies that D is pairwise aligned because all agents strictly prefer any coalition with strictly positive output to any coalition with zero output, and are indifferent between any two coalitions with zero output (because $D_{a,C}(y) = 0$ iff $y = 0$ for every $C \in \mathcal{C}$, $a \in C$).

5.1. *Pareto-Efficient Sharing Rules: A Characterization*

A sharing rule is *Pareto-efficient* if $\sum_{a\in C} D_{a,C}(y) = y$ for any $C \in \mathcal{C}$ and $y \geq 0$. The equal sharing and Nash bargaining are efficient. Efficient pairwise-aligned sharing rules may be characterized as follows.

PROPOSITION 1: *Suppose that the family of coalitions $\mathcal{C}$ satisfies* C1 *and* C2, *and the functions $D_{a,C}$ are strictly increasing and continuous*, *and* $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$ *for all $C \in \mathcal{C}, a \in C$. The sharing rule D is pairwise aligned and efficient if and only if there exist increasing*, *differentiable*, *and strictly log-concave functions $U_a : R_+ \to R_+$, $a \in A$, such that $\frac{U_a}{U_a'}(0) = 0$ and*

$$(D_{a,C}(y))_{a\in C} = \arg\max_{\sum_{a\in C} s_a \leq y} \prod_{a\in C} U_a(s_a), \quad y \in R_+, C \in \mathcal{C} - \{A\}.$$

In the proposition, we allow $U_a'(0) = +\infty$. We refer to functions U_a as *bargaining functions*. In Nash bargaining these are simply agents' utility (or profit) functions. An inspection of the proof shows that the representation remains true if C equals the grand coalition A provided that $A \in \mathcal{C}$ and agents' preferences are additionally pairwise aligned over the grand coalition. In what follows, we use the term *regular sharing rule* to refer to sharing rules which are aligned over the grand coalition, and such that all functions $D_{a,C}$ are strictly increasing and continuous, and $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$. In addition, as illustrated at the end of the next subsection, the main thrust of the result remains true when we drop the efficiency assumption.[18]

The above results imply the following corollary

COROLLARY 2: *Suppose that the family of coalitions $\mathcal{C}$ satisfies* C1–C3 *and the sharing rule D is regular. There is a stable coalition structure for each preference profile induced by the sharing rule if and only if there exist increasing, differentiable, and strictly log-concave functions $U_a : R_+ \to R_+$, $a \in A$, such that $\frac{U_a}{U_a'}(0) = 0$ and*

$$(D_{a,C}(y))_{a\in C} = \arg\max_{\sum_{a\in C} s_a \le y} \prod_{a\in C} U_a(s_a), \quad y \in R_+, C \in \mathcal{C} - \{A\}.$$

As an illustration of the above characterization results, consider a linear sharing rule

$$D_{a,C}(y) = d_{a,C}\, y,$$

where shares $d_{a,C}$ are positive constants such that $\sum_{a\in C} d_{a,C} = 1$. Corollary 1 implies that the linear sharing rule admits stable coalition structures in all states of nature if and only if the shares $d_{a,C}$ satisfy the proportionality condition

$$\frac{d_{a,C}}{d_{b,C}} = \frac{d_{a,C'}}{d_{b,C'}}$$

for all $C, C' \neq A$ and $a, b \in C \cap C'$.[19] This observation may be rephrased in terms of Nash bargaining. Nash bargaining leads to linear sharing of value if agents' utilities are $U_a(s) = s^{\lambda_a}$ for some agent-specific constants (bargaining powers) λ_a. The resulting shares satisfy the above proportionality condition, and any profile of shares $\{d_{i,C}\}_{C\in\mathcal{C}-\{A\}}$ that satisfies the proportionality condition may be interpreted as generated by Nash bargaining. Thus, a profile of shares $\{d_{i,C}\}_{C\in\mathcal{C}-\{A\}}$ guarantees the existence of stable matching for all $\mathbf{y} : \mathcal{C} \to R_+$ if and only if the shares can be represented as an outcome of Nash bargaining.

[18]Proposition 1 extends the main insight of Lensberg (1987) to many-to-one matching and other environments satisfying C1 and C2. Lensberg constructed a representation resembling that of Proposition 1 for consistent and efficient sharing rules; consistency and pairwise alignment of sharing rules are closely related as discussed in footnote 5. The results are logically independent even in a one-sided coalition formation setting.

[19]In the context of one-sided coalition formation, Banerjee, Konishi, and Sönmez (2001) proved a slightly weaker variant of one of the above implications: assuming the proportionality condition for all coalitions C, C', they showed that a stable coalition structure exists. The converse implication is new.

5.2. *Assortative Matching and Coalition Formation*

When shares are divided by a stability-inducing sharing rule, agents sort themselves into coalitions in a predictably assortative way. Let us start by looking at Nash bargaining in which each agent a is endowed with an increasing, concave, and differentiable utility function U_a normalized so that $U_a(0) = 0$. In this setting, agents sort themselves into coalitions according to their fear of ruin and their productivity. Recall that Aumann and Kurz's (1977) fear-of-ruin coefficient is defined as $\frac{U_a(s)}{U_a'(s)}$. We say that agent a has weakly higher fear of ruin than agent b if $\frac{U_a(s)}{U_a'(s)} \geq \frac{U_b(s)}{U_b'(s)}$ for all $s > 0$, and agent a has strictly higher fear of ruin if strict inequality holds for all $s > 0$.

We assume that each agent $a \in A$ is endowed with productivity type θ_a from a space Θ of types, and the output $\mathbf{y}(C)$ is fully determined by the size of C and productivity types of members of C. In particular, if C_a and $C_b = C_a \cup \{b\} - \{a\}$ are coalitions and $\theta_a = \theta_b$, then $\mathbf{y}(C_a) = \mathbf{y}(C_b)$. We further assume that the space Θ is endowed with a partial ordering on types, and that the output is strictly increasing in the partial ordering on $\theta_a \in \Theta$ (keeping productivity types of agents in $C - \{a\}$ constant). Finally, we assume that the family of coalitions is symmetric, that is, for any agents a and b who are on the same side of the market, if a coalition C contains a but not b, then $(C \cup \{b\}) - \{a\}$ is a coalition. We say that two agents are on the same side of the market if both are workers or if both are firms.

In this environment, the resulting stable coalition structure is assortative: agents with high productivity and high fear of ruin form coalitions together. Proposition 2 formalizes this statement.

PROPOSITION 2: *Assume that the family of coalitions is symmetric and that the outputs are increasing in agents' productivity. Let C_1 and C_2 belong to the same stable coalition structure, and let agents $a_1, b_1 \in C_1$ and $a_2, b_2 \in C_2$ be such that a_1 and a_2 are on the same side of the market and b_1 and b_2 are on the same side of the market. If a_1 is weakly more productive and has weakly higher fear of ruin than a_2, with at least one of the relations being strict, then it is not possible that b_2 is weakly more productive and has weakly higher fear of ruin than b_1, with at least one of the relations being strict.*

PROOF: The proof is by a straightforward indirect argument. Assume that there are coalitions C_1, C_2 and agents a_1, a_2, b_1, b_2 that falsify the proposition. Recall from the analysis of Nash bargaining in Section 2 that the fear of ruin coefficient χ takes a common value χ_{C_1} for all agents in C_1 and a common value χ_{C_2} for all agents in C_2, and that agents always prefer coalitions with higher χ. Because of the symmetry between assumptions on C_1 and C_2, we may assume that $\chi_{C_1} \geq \chi_{C_2}$. Moreover, by symmetry of the family of coalitions, $C = (C_1 - \{b_1\}) \cup \{b_2\}$ is a coalition. Since b_2 is more productive and more risk averse than b_1, with at least one of the relations being strict, we must have $\chi_C > \chi_{C_1}$. Because agents' preferences over coalitions are aligned with χ, all agents in $C \cap C_1$ prefer it to C_1 and agent b_1 prefers C to C_2. Thus, C would be a blocking coalition, a contradiction. *Q.E.D.*

As an example of an application of Proposition 2, consider the case of agents endowed with identical utility functions who differ in their productivity types $\theta_a \in \Theta$. Such agents sort themselves in terms of productivity. This case of Proposition 2 generalizes the main result of Farrell and Scotchmer (1988).[20]

As another example, consider the case when $\mathbf{y}(C)$ depends on C only through its cardinality $|C|$ and agents have power utilities $U_a(s) = s^{\lambda_a}$. Then the coefficients λ_a can be interpreted as agents' bargaining powers and the result says that agents sort themselves on bargaining powers: for any two coalitions in the stable coalition structure, the largest bargaining power of a worker in one of them is weakly lower than the smallest bargaining power of a worker in the other coalition and similarly for firms. In particular, the assortative structure implies that the differences among bargaining powers of agents within a coalition of a stable coalition structure are suppressed relative to the differences of bargaining powers among all agents.

Provided the coalition structure satisfies C1–C3, the assortative structure of Proposition 2 remains true for all stability-inducing and efficient sharing rules characterized in Corollary 1 and Proposition 1. Proposition 1 allows us to define analogs of U_a for all such sharing rules, and the above proof of Proposition 2 extends to the more general case without any changes. Furthermore, the assortative structure remains true for all stability-inducing sharing rules if—in addition to agents' impact on productivity—we explicitly account for agents' impact on the inefficiency. This is so because the inefficient sharing rules can be viewed as dividing the effective output (defined as the sum of agents shares) in an efficient way. Proposition 1 allows us thus to find the bargaining functions for the induced efficient sharing rule and, productivity and inefficiency factors held constant, agents sort themselves according to the fear-of-ruin coefficient of their bargaining functions. Finally, if the fear-of-ruin coefficients of two bargaining functions U_a and U_b are not comparable because the relationship between $\frac{U_a}{U_a'}(s)$ and $\frac{U_b}{U_b'}(s)$ depends on the stake s, then agents still sort themselves according to their fear of ruin calculated at relevant stakes.

[20]Farrell and Scotchmer analyzed the equal-sharing rule (which corresponds to assuming all agents are endowed with identical utility functions) and imposed a linear relationship between outputs and one-dimensional productivity types.

5.3. *Comparative Statics: How Equality Among Members of a Coalition Increases the Chances the Coalition Is Stable*

This subsection examines how the probability that any given coalition L is in a stable coalition structure depends on the sharing rule.[21] We assume that outputs $\mathbf{y}(C)$ are independently drawn from absolutely continuous log-concave distributions on R_+. Log concavity means that the logarithm of the cumulative distribution function (c.d.f.) is concave and is equivalent to a monotone hazard rate condition. Many of the distributions studied in economics, including the uniform distribution on $[0, 1]$ and the exponential distribution, satisfy this property. We allow the distribution of $\mathbf{y}(C)$ to depend on C through its cardinality $|C|$ but not otherwise. Finally, we impose the following symmetry assumption on the family of coalitions: for any workers a and b, if a coalition C contains a but not b, then $(C \cup \{b\}) - \{a\}$ is a coalition.

Under these assumptions, the equal-sharing rule maximizes the probability the grand coalition A is stable. An analogous claim holds true for any other coalition $L \neq A$, but its formulation requires some care. There are two forces that increase the probability that $L \neq A$ belongs to a stable coalition structure: equality among workers in L and the relative bargaining strength of members of L when compared to other agents. The probability of L being stable is maximized as we approach the limit in which workers in L share the output equally, but would get nothing in any coalition containing an agent from $A - L$. Since no regular sharing rule accomplishes this limit, we prove the maximal property of equal sharing while controlling for the relative strengths of members of L vis-à-vis other agents.

PROPOSITION 3: *Assume that the family of coalitions $\mathcal{C}$ is symmetric and satisfies* C1–C3, *that L is a coalition*, *and that outputs $\mathbf{y}(C)$ are independently drawn from size-dependent log-concave distributions on R_+. There is a partition of the class of stability-inducing*, *efficient*, *and regular sharing rules such that each element of the partition contains a unique sharing rule D that equalizes shares of workers in L for all output levels*, *and the probability of L being stable under D is weakly higher than under any other sharing rule from the element of the partition* (*and strictly higher if the distributions are strictly log concave*). *In particular*, *if $L = A$*, *then the equal-sharing rule maximizes the probability of A being stable among all stability-inducing*, *efficient*, *and regular sharing rules*.

The proof of Proposition 3 allows us to construct a partial ordering on all stability-inducing efficient sharing rules, such that the probability of L being

[21] We study regular sharing rules. For such rules, the probability of a particular coalition structure is well defined when $\mathbf{y}(C)$ are drawn from continuous distributions because Theorem 1 implies that the stable coalition structure is generically unique when the sharing rule is regular, and it is easy to extend the argument to conclude that in the current setting it is unique with probability 1.

stable is increasing in the partial ordering. In lieu of a heuristic of the proof, let us look at such an ordering, restricting attention to the following class of linear sharing rules: each agent a is endowed with bargaining power $\lambda_a > 0$, and coalition C divides output y so that the share of agent a is

$$D_{a,C}(y) = \frac{\lambda_a}{\sum_{b \in C} \lambda_b} y.$$

The probability of coalition L forming is then decreasing in inequality among bargaining powers of workers in L. Denote by $\lambda_{(i)}$ the ith highest value of λ_w among workers $w \in L$. We keep the bargaining powers of firms and of agents not in L proportional to $\sum_{w \in L \cap W} \lambda_w$ and assess the inequality among workers in L with the partial order

$$(\lambda_w)_{w \in L \cap W} \geq (\lambda'_w)_{w \in L \cap W}$$
$$\iff \quad \frac{\lambda_{(i)}}{\lambda_{(i+1)}} \geq \frac{\lambda'_{(i)}}{\lambda'_{(i+1)}}, \quad i = 1, \ldots, |L \cap W|;$$

the ordering is strict if at least one of the above inequalities is strict.

PROPOSITION 4: *Assume that the family of coalitions is symmetric and that outputs* $\mathbf{y}(C)$ *are independently drawn from size-dependent log-concave distributions on* R_+. *Then the probability that the coalition L is stable is decreasing in the above-defined partial order on the profiles of bargaining powers. The probability is strictly decreasing if output distributions are strictly log-concave on* R_+.

Let us sketch the proof for the case of $L = A$. The Appendix gives omitted parts of the argument. First note that rescaling all bargaining powers by a constant changes neither the ordering nor agents' payoffs, and hence we may assume that $\sum_{a \in A} \lambda_a = \sum_{a \in A} \lambda'_a$. Assume that there are two workers, say a and b, whose bargaining powers differ, $\lambda_a < \lambda_b$. Take any coalition C. If $a, b \in C$ or $a, b \notin C$, then a small increase in λ_a and an offsetting decrease in λ_b that keeps the sum of the two powers constant does not change the probability that A is blocked by C. If $a \in C$ but $b \notin C$, then $C \cup \{b\} - \{a\}$ is a coalition by the symmetry of the family of coalitions, and we use the log-concavity of the distributions to show that the above adjustment of the two bargaining powers decreases the joint probability that A is blocked. Hence, the above adjustment of bargaining powers increases the product of probabilities that A is not blocked by any coalition and, hence, the probability that A is stable. To conclude the proof, we then show that if one profile of bargaining powers is dominated by another in the above partial ordering, then there is a finite sequence of adjustments of bargaining powers that connects the two.

An analogous argument shows that the more equal the sharing among a group of workers, the more likely they are to be in the same coalition in the stable matching or coalition structure.

6. APPLICATIONS: MATCHING

6.1. *Complementarities in Matching*

The results of the preceding sections are applicable to many-to-one matching situations with complementarities and peer effects. In the example of Section 2, we have seen that in state ω_2, firm g but not firm f treats workers 1 and 2 as complementary both under equal sharing and Nash bargaining, and that each worker cares whether the other one works for the same firm. In general, under equal sharing or Nash bargaining, the firm's preferences may treat two or more workers as complementary depending on the profile of outputs $\mathbf{y}$. The peer effects are inherent to both equal sharing and Nash bargaining: workers care about which other workers belong to their coalition.

The presence of complementarities means that some of the standard comparative statics derived in the theory of many-to-one matching under the standard assumptions of gross substitutes and absence of peer effects no longer hold true.[22] A major standard comparative static result says that removing an agent from one side of the market weakly increases the payoffs of the other agents on the same side of the market and weakly decreases payoffs of agents on the other side (Crawford (1991)). In contrast, even if we assume that agents' payoffs are determined in equal sharing or in Nash bargaining and that the stable matching is unique, removing a worker may lead to a change of the stable matching that results in some firms obtaining higher payoffs and some workers obtaining lower payoffs. For instance, consider equal sharing in state ω_2 of the example in Section 2. The unique stable matching is $\{\{f\}, \{g, 1, 2\}\}$ when all agents are available and $\{\{f, 2\}, \{g\}\}$ when worker 1 is not available. Thus, firm f benefits and worker 2 loses when worker 1 is removed. Similarly, removing a firm may increase the payoffs of some workers and decrease the payoff of some firms as illustrated below.

EXAMPLE 2: Consider three firms f_1, f_2, f_3 and two workers w_1, w_2. Assume that agents share outputs equally, and that the outputs are such that

$$\mathbf{y}(\{f_1, w_1\}) = 2, \quad \mathbf{y}(\{f_2, w_2\}) = 1, \quad \mathbf{y}(\{f_3, w_1, w_2\}) = 2,$$

[22] We consider only many-to-one matchings that satisfy C2: any firm and any two workers can form a coalition together. For the purposes of our discussion, this is not a strong restriction, as complementarities and peer effects can be present only in many-to-one matching situations in which some firms can be matched with two (or more) workers. By Theorem 1, the stable matching is unique in our setting if we adapt the standard assumption that agents' preferences are strict. The uniqueness implies that the lattice structure of the set of stable matchings (Gale and Shapley (1962)) and the so-called rural hospital theorem (Roth (1984)) are true in our setting. As usual, the properties are not true when indifference is allowed.

and all other outputs equal zero. The unique stable matching is $\{\{f_1, w_1\}, \{f_2, w_2\}, \{f_3\}\}$ when all agents are available and $\{\{f_2\}, \{f_3, w_1, w_2\}\}$ when firm f_1 is not available. Thus, removing firm f_1 decreases the payoff of firm f_2 and increases the payoff of worker w_2.

Weak versions of the standard comparative statics remain true: it is straightforward to check that adding a worker weakly improves the payoff for at least one firm and adding a firm weakly improves the payoff for at least one worker.

6.2. *Implementation and Strategic Play*

Under the substitutes and absence-of-peer-effects conditions, the Gale and Shapley (1962) deferred acceptance algorithm produces a stable coalition structure in a many-to-one matching provided agents act truthfully. However, agents have incentives to be strategic and in equilibrium, the outcome of the deferred acceptance algorithm does not need to be stable (Dubins and Freedman (1981), Roth (1982), and Ma (2010)). At the same time, there are non-cooperative games that implement the core stable correspondence (Kara and Sönmez (1997)). The stability of the outcome of coalition formation thus depends on the details of the non-cooperative game agents play.[23] The following result shows that under pairwise alignment, the details of the process of coalition formation are less important.

PROPOSITION 5: *Consider a non-cooperative game (extensive or normal form) among agents from A and a mapping $\hat{\mu}$ from agents' strategies Σ to coalition structures such that for each coalition $C \in \mathcal{C}$ there is a profile of strategies σ_C of agents in C such that $C \in \hat{\mu}(\sigma)$ for all strategy profiles $\sigma \in \Sigma$ that agree with σ_C on C. If the family of coalitions satisfies* C1 *and* C2, *and agents' preferences come from a rich domain of pairwise-aligned preference profiles, then for every stable coalition structure μ, there is a Strong Nash Equilibrium σ such that $\mu = \hat{\mu}(\sigma)$.*[24,25]

[23]The non-cooperative game agents play is less important in one-to-one matching as shown by Ma (1995), Shin and Suh (1996), and Sönmez (1997). One-to-one matching satisfies both the pairwise-alignment condition and the substitutes and absence-of-peer-effects conditions.

[24]We are implicitly assuming that each agent's payoff is uniquely determined by the coalition in $\hat{\mu}(\sigma)$ that the agent belongs to. A profile of players' strategies σ is a strong Nash equilibrium if there does not exist a subset of players that can improve the payoffs of all its members by a coordinated deviation, while players not in the subset continue to play strategies from σ (Aumann (1959)). Similar results are true for the strong perfect equilibrium of Rubinstein (1980) and the coalition-proof Nash equilibrium of Bernheim, Peleg, and Whinston (1987).

[25]In Proposition 5, we can replace the pairwise alignment and assumptions C1 and C2 with the assumption that there are no cycles in agentsŠ preferences over proper coalitions. Recall that when the family of coalitions satisfies C1–C3 and the domain of preferences is rich, the pairwise alignment is equivalent to the no-cycle assumption. Niederle and Yariv (2009) further develop the theory of games implementing stable one-to-one matchings when preferences satisfy the no-cycle assumption.

The coalition formation game based on the Gale and Shapley deferred acceptance algorithm satisfies the assumption of Proposition 5 as does the single-round-of-application game in which each worker applies for one or no jobs and then each firm selects its workforce from among its applicants.

Proposition 5 relies on the alignment of agents' preferences but not on the many-to-one structure. The converse claim that any strong Nash equilibrium gives a stable coalition structure is straightforward and does not require any assumptions on preferences. Thus, the proposition implies that when agents' preferences are aligned, then the core correspondence is implemented in strong Nash equilibrium (and hence partially implemented in Nash equilibrium) of any game from the broad class of non-cooperative games described in the proposition.

7. CONCLUDING REMARKS

We have seen which sharing rules and, more generally, which preference domains guarantee the existence of stable coalition structures, and we have analyzed properties of such sharing rules and preference domains. Let us conclude by a preliminary look at how the results of this paper can be applied toward market design and the study of holdup.

7.1. *Sharing Rules as an Instrument of Market Design*

Having established the importance of stability, the literature on the design of matching markets that follows Roth (1984) has been primarily concerned with algorithms used in the matching process. The algorithm used in the centralized matching is a primary tool to achieve stability, but it is not the only one.

For instance, consider the well studied environment in which stability matters: the matching between residents and U.S. hospitals described by Roth (1984) and Roth and Peranson (1999). The matching is organized by the National Resident Matching Program (NRMP), which represents several medical, medical education, and medical student organizations. The NRMP plays the role of a social planner and wants the resulting matching to be stable because the lack of stability has historically led to the unraveling of the resident–hospital matching process. The main instrument used by the NRMP to achieve stability is the matching algorithm. However, the medical and medical education organizations that formed NRMP also regulate the residency system in other ways; the regulations influence residents' and hospitals' payoffs, and they affect the stability of the matching. For instance, through the Accreditation Council for Graduate Medical Education, the organizations recently enacted regulations limiting residents' working hours. The regulations affected payoffs both for residents and for hospitals' faculty.[26] The regulations resemble sharing rules, or preference domains, in that they influence the mapping from states of nature—which are ex ante unknown to the medical organizations—to the payoffs of match participants. While our model lacks the institutional detail required for direct application to residency matching, its results may be viewed as a step toward understanding what tools—other than the matching algorithm—may be employed to achieve stability in matching markets.

7.2. *Holdup*

Inflexible sharing of output leads to holdup in coalition formation. For instance, consider the setting in which agents share output in Nash bargaining with constant bargaining powers. An agent may be better off with a lower rather than higher bargaining power—other things held equal—when a low bargaining power allows him or her to form a highly productive coalition, while a high bargaining power makes formation of such a productive coalition impossible by lowering the payoffs of its other members below their outside options. In a related matter, the stable coalition structure does not necessarily maximize the sum of agents' payoffs. Both of these problems illustrate the holdup inherent in any model in which the anticipated sharing of output is fixed at the matching stage and agents cannot make side payments or contract on them when forming coalitions.

The holdup caused by inflexible sharing of payoffs is far from a solely theoretical possibility. For instance, Baker, Gibbons, and Murphy (2008) reported that interviews with practitioners involved in the formation of alliances (coalitions) among firms led them to conclude that the lack of flexibility in dividing payoffs that accrue directly to firms in an alliance—rather than to the alliance itself—is one of two main factors determining the form and performance of alliances (the second main factor being governance structure). In what they heard from practitioners, the inflexible sharing of payoffs played a markedly larger role than the inadequate specific investments identified as a source of holdup by Grossman and Hart (1986) and Hart and Moore (1990), and studied by the rich literature on the theory of the firm.

APPENDIX: PROOFS

LEMMA 1: *Let $\succsim_A$ be a preference profile such that the coalition structure $\mu \neq \{A\}$ is stable and let $\succsim'_A$ be a preference profile such that*

$$C \succsim'_a C' \quad \Longleftrightarrow \quad C \succsim_a C'$$

[26]The majority of residents surveyed by Niederee, Knudtson, Byrnes, Helmer, and Smith (2003), and Brunworth and Sindwani (2006) supported the restriction, while the majority of teaching hospitals' faculties opposed it.

for $a \in C, C' \in \mathcal{C} - \{A\}$. *Then either* $\{A\}$ *or* μ *is stable with respect to* $\succsim'_A$. *Moreover, if* μ *is the unique* $\succsim_A$*-stable coalition structure and* $C \succsim_a A$ *for any* $a \in C \in \mathcal{C}$, *then there are no* $\succsim'_A$*-stable coalition structures other than* μ *and* $\{A\}$.

PROOF: Take any $\succsim_A$-stable coalition structure μ. Let $\mu(a)$ denote the coalition of agent a in coalition structure μ. Because $\succsim'_A$ is equivalent to $\succsim_A$ on $\mathcal{C} - \{A\}$, no coalition other than A can $\succsim'_A$-block μ. Thus, either μ is $\succsim'_A$-stable or is $\succsim'_A$-blocked by A. In the latter case,

$$A \succ'_a \mu(a) \quad \text{for} \quad a \in A.$$

Now, if $\{A\}$ were not $\succsim'_A$-stable, then there would be a coalition $C \neq A$ such that

$$C \succ'_a A \quad \text{for} \quad a \in C.$$

The two displayed preferences would then imply that $C \succ'_a \mu(a)$ and hence $C \succ_a \mu(a)$ for $a \in C$, contrary to stability of μ. Thus, if μ is not $\succsim'_A$-stable, then A is. The uniqueness claim is straightforward. *Q.E.D.*

LEMMA 2: *If there are no n-cycles for any* $n = 3, 4, \ldots,$ *then there is a stable coalition structure*.[27]

PROOF: By Lemma 1, to show that lack of n-cycles for any $n = 3, 4, \ldots$ implies that there is a stable coalition structure, it is enough to prove this claim under the additional assumption that either $A \notin \mathcal{C}$ or A is the $\precsim_A$-worst choice for each agent. We proceed by induction with respect to $|A|$. For $|A| = 1$, the claim is true. For the inductive step, assume that the claim is true whenever the number of agents is less than $|A|$. By way of contradiction, let us also assume that there is no stable coalition structure on A. Then, for any coalition $C \in \mathcal{C}$, there must exist a proper coalition C' that blocks C, that is, $C \cap C' \neq \emptyset$ and all agents $a \in C \cap C'$ strictly prefer C' to C. Indeed, if there were a coalition C

[27]The proof shows something more: if there are no n-cycles, $n = 3, 4, \ldots,$ in which all agents in $C_k \cap C_{k+1}$ strictly prefer C_{k+1} to C_k, then there is a stable coalition structure. Also, since lack of n-cycles implies lack of k-cycles for all $k \leq n$, Lemma 2 shows that if there are no n-cycles for odd integers $n \geq 3$, then there is a stable coalition structure. For the roommate problem, Lemma 2 follows from Chung's (2000) "no-odd-rings" condition; however, his proof relies on the structure of the roommate problem. Lemma 2 also strengthens a result from Farrell and Scotchmer (1988): they assumed the existence of a complete ordering on all coalitions that satisfy an equivalence counterpart of implication (3) from the proof of Lemma 5 below, and showed that there exists a stable coalition structure. Banerjee, Konishi, and Sönmez (2001) relaxed the Farrell and Scotchmer ordering condition to a requirement that among the coalitions formed by any subset of agents, there is a "top" coalition that is preferred by its members to all alternatives. A relaxed version of their top coalition property is true in our setting as shown in the proof of Proposition 5. Their result is logically independent of Lemma 2.

that is not blocked, then the following coalition structure would be stable: C and coalitions that form a stable structure on $A - C$. Hence every coalition can be blocked. Let us thus start with coalition C_1 and find a proper coalition C_2 that blocks C_1, then find a proper coalition C_3 that blocks C_2, and so forth. Since there is a finite number of coalitions in $\mathcal{C}$ and all are proper, there is an n-cycle. Moreover, n must be larger than 2, as C_2 cannot be blocked by C_1. This contradiction completes the proof. *Q.E.D.*

LEMMA 3: *Let $\mathcal{C}$ satisfy* C1 *and* C2, *and let* $\mathbf{R}$ *satisfy* R1. *If all profiles in* $\mathbf{R}$ *are pairwise aligned*, *then no profile in* $\mathbf{R}$ *admits a* 3-*cycle*.

PROOF: By way of contradiction, assume that there are proper coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ and agents a_1, a_2, a_3 such that

$$C_{3,1} \prec_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}.$$

Assumption C1 and pairwise alignment imply that at least two of the agents a_1, a_2, a_3 are workers, and then C2 implies the existence of an agent a_0 and proper coalitions $C'_{1,2}, C'_{2,3}, C'_{3,1}$ such that $C'_{k,k+1} \ni a_k, a_{k+1}$ and $a_0 \in C'_{1,2} \cap C'_{2,3} \cap C'_{3,1}$.

If $C'_{1,2} = C'_{2,3}$, then we can assume that $C'_{1,2} = C'_{2,3} = C'_{3,1} = C'$. We obtain a contradiction in the same way as in the analysis of the Kalai–Smorodinsky example presented after the statement of Theorem 1.

If $C'_{1,2}, C'_{2,3}$, and $C'_{3,1}$ are all different, then R1 implies that there is a pairwise-aligned profile $\precsim'_A$ such that

$$C_{3,1} \precsim'_{a_1} C'_{1,2} \precsim'_{a_1} C_{1,2}, \qquad (2)$$
$$C_{1,2} \precsim'_{a_2} C'_{2,3} \precsim'_{a_2} C_{2,3},$$
$$C_{2,3} \precsim'_{a_3} C'_{3,1} \precsim'_{a_3} C_{3,1},$$

and all agents' $\precsim'_A$-preferences between pairs of coalitions not including $C'_{1,2}$, $C'_{2,3}$, and $C'_{3,1}$ are the same as their $\precsim_A$-preferences. Pairwise alignment of $\precsim'_A$ gives $C'_{3,1} \precsim'_{a_1} C_{3,1}$ and thus

$$C'_{3,1} \precsim'_{a_1} C'_{1,2}.$$

Similarly

$$C'_{1,2} \precsim'_{a_2} C'_{2,3} \quad \text{and} \quad C'_{2,3} \precsim'_{a_3} C'_{3,1}.$$

Because agent a_1 strictly prefers $C_{1,2}$ over $C_{3,1}$, at least one of the preference relations in (2) must be strict and thus at least one preference relation above is strict. Hence, the pairwise alignment implies that

$$C'_{3,1} \precsim'_{a_0} C'_{1,2} \precsim'_{a_0} C_{2,3} \precsim'_{a_0} C'_{3,1},$$

with at least one preference relation strict, which is a contradiction. *Q.E.D.*

LEMMA 4: *Let $\mathcal{C}$ satisfy* C1, *let* $\mathbf{R}$ *satisfy* R1, *and let* $n \in \{3, 4, \ldots\}$. *If no profile in* $\mathbf{R}$ *admits a* 3-*cycle*, *then no profile in* $\mathbf{R}$ *admits an n-cycle*.

PROOF: For an inductive argument, fix $m \geq 4$ and assume that preferences in $\mathbf{R}$ do not admit n-cycles for $n = 3, \ldots, m-1$. We want to show that preferences in $\mathbf{R}$ do not admit m-cycles. By way of contradiction, assume that there is $\precsim_A \in \mathbf{R}$ that admits an m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} \cdots \precsim_{a_m} C_{m,1}$. Then a_1 or a_2 is a worker, as otherwise C1 implies that $a_1 = a_2$ and

$$C_{m,1} \precsim_{a_1} C_{2,3} \precsim_{a_3} \cdots \precsim_{a_m} C_{m,1}$$

is an $(m-1)$-cycle, contrary to the inductive assumption. By symmetry, we may assume that $a_1 \in W$. Assumption C1 then implies that there is C such that $a_1, a_3 \in C$. We consider two cases.

Case $C = C_{i,i+1}$ for some $i = 1, \ldots, m$: Look at $C_{1,2}, C_{2,3}, C$ and conclude from the lack of 3-cycles that one of the following three subcases would obtain.

- $C_{1,2} \prec_{a_1} C = C_{i,i+1}$. Then $C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} \cdots \precsim_{a_m} C_{m,1} \precsim_{a_1} C_{i,i+1}$ and the last preference is strict because $C_{m,1} \precsim_{a_1} C_{1,2} \prec_{a_1} C = C_{i,i+1}$. For the same reason, $i \neq 1, m$. Thus, there would be an $(m-i+1)$-cycle, contrary to the inductive assumption.
- $C_{2,3} \succ_{a_3} C = C_{i,i+1}$. Then $C_{i,i+1} \precsim_{a_3} C_{3,4} \precsim_{a_4} \cdots \precsim_{a_i} C_{i,i+1}$ and the first preference is strict because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$. For the same reason, $i \neq 2, 3$. Thus, there would be an n-cycle with $n = i - 2 \bmod m$, contrary to the inductive assumption.
- $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$. Then $C \precsim_{a_3} C_{3,4} \precsim_{a_4} \cdots \precsim_{a_m} C_{m,1} \precsim_{a_1} C$ with at least one strict preference inherited from the m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} \cdots \precsim_{a_m} C_{m,1}$. Thus, there would be an $(m-1)$-cycle, contrary to the inductive assumption.

Case $C \neq C_{i,i+1}$ for all $i = 1, \ldots, m$: By R1, there is a pairwise-aligned preference profile $\precsim'_A \in \mathbf{R}$ such that all preferences along the m-cycle are preserved and

$$C_{m,1} \precsim'_{a_1} C \precsim'_{a_1} C_{1,2}.$$

We cannot have $C \prec'_{a_3} C_{2,3}$ for then $C \prec'_{a_3} C_{2,3} \precsim'_{a_3} C_{3,4}$ and hence $C \prec'_{a_3} C_{3,4} \precsim'_{a_4} \cdots \precsim'_{a_m} C_{m,1} \precsim'_{a_1} C$ would be an $(m-1)$-cycle. Thus, $C \succsim'_{a_3} C_{2,3}$, and

$$C \precsim'_{a_1} C_{1,2} \precsim'_{a_2} C_{2,3} \precsim'_{a_3} C.$$

The lack of 3-cycles implies that all agents above are indifferent. But then $C \precsim'_{a_3} C_{3,4} \precsim'_{a_4} \cdots \precsim'_{a_m} C_{m,1} \precsim'_{a_1} C$ would be an $(m-1)$-cycle with at least one strict preference inherited from the m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} \cdots \precsim_{a_m} C_{m,1}$. This contradiction completes the proof. *Q.E.D.*

LEMMA 5: *If preference profile $\precsim_A$ has no n-cycles for $n = 2, 3, \dots$, and agents' preferences are strict, then there is at most one stable coalition structure different than $\{A\}$.*

PROOF: Let us define a partial ordering $\trianglelefteq$ on proper coalitions as follows: $C \trianglelefteq C'$ iff there exists a sequence of proper coalitions $C_{i,i+1} \in \mathcal{C}$ such that $C = C_{1,2}$, $C' = C_{m,m+1}$, and for each $i = 2, \dots, m$, there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ that weakly prefers $C_{i,i+1}$ to $C_{i-1,i}$. The relation is strict if for at least one $i = 2, \dots, m$, all agents $a \in C_{i-1,i} \cap C_{i,+1}$ strictly prefer $C_{i,i+1}$ to $C_{i-1,i}$. Notice that all proper coalitions C, C' with a nonempty intersection are comparable and

$$\text{(3)} \qquad \text{if} \quad C \trianglelefteq C', \quad \text{then} \quad C \precsim_a C' \quad \text{for all } a \in C \cap C'.$$

The relation is transitive by construction. It is also acyclic in the sense of preference theory: given transitivity, a relation is acyclic if there are no coalitions C and C' such that $C \trianglelefteq C'$ and $C \triangleright C'$.

To prove the uniqueness claim, first consider the case $A \notin \mathcal{C}$. Let $C_1, C_2, \dots, C_k$ be maximal coalitions in ordering $\trianglelefteq$. By lack of 2-cycles and preference strictness, the coalitions $C_1, \dots, C_k$ are disjoint. By (3) and strictness of preferences, the maximal coalitions must be a part of every stable coalition structure, and by induction, there is a unique stable coalition structure. Lemma 1 completes the argument for the case $A \in \mathcal{C}$. *Q.E.D.*

PROOF OF THEOREM 1: With regard to the proof of Theorem 1, the existence claim follows from Lemmas 3, 4, and 2. The uniqueness claim follows from Lemma 5. *Q.E.D.*

LEMMA 6: *Let* **R** *satisfy* R2. *If all profiles $\precsim_A \in \mathbf{R}$ admit a stable coalition structure, then there is no $\precsim_A \in \mathbf{R}$ and* 3*-cycle $C_{3,1} \prec_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}$ such that $C_{i-1,i} \cap C_{i,i+1} = \{a_i\}$ for $i = 1, 2, 3$.*

PROOF: By way of contradiction, assume that there exist $\precsim_A \in \mathbf{R}$ and 3-cycle of coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ prohibited by the lemma. Notice that coalitions $C_{i,i+1}$ are all different because if $C_{i-1,i} = C_{i,i+1}$, then $C_{i-1,i} = C_{i,i+1} = \{a_i\}$, and hence $a_1 = a_2 = a_3$, and this agent's preferences would be cyclic. We modify the preference profile and construct a profile in **R** that does not admit a stable coalition structure. At each step of the procedure let us continue to denote the current profile by $\precsim_A$.

First, use R2(ii) with $C = C_{1,2}$ to find a preference profile $\precsim_A \in \mathbf{R}$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \precsim_{a_3} C_{3,1}$. Then use R2(ii) with $C = C_{2,3}$ to find $\precsim_A$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Last, one by one, for all coalitions C that (i) contain an agent from $C_{1,2} \cup C_{2,3} \cup C_{3,1}$ and (ii) are different than $C_{1,2}, C_{2,3}, C_{3,1}$, use R2(i) to find $\precsim_A \in \mathbf{R}$ such that $C \prec_a C_{k,k+1}$ for $a \in C \cap C_{k,k+1}$, $k = 1, \dots, 3$.

The resulting profile of preferences belongs to **R** and does not admit a stable coalition structure. This completes the proof. *Q.E.D.*

LEMMA 7: *Assume that* **R** *satisfies* R1 *and no profile in* **R** *admits a* 3-*cycle prohibited by Lemma* 6. *If agents* a, b, c *and coalitions* C, C', C_a, C_b *are such that*

$$C_a \cap C = C_a \cap C' = \{a\},$$
$$C_b \cap C = C_b \cap C' = \{b\},$$
$$C_a \cap C_b = \{c\},$$

then $C \precsim_a C' \Longrightarrow C \precsim_b C'$ *for all* $\precsim_A \in \mathbf{R}$.

PROOF: By way of contradiction, assume that

$$C \precsim_a C' \quad \text{and} \quad C' \prec_b C.$$

Because $C_a \neq C_b$, the condition R1 implies that there is $\precsim_A \in \mathbf{R}$ (we continue using the symbol $\precsim_A$ for the new profile) such that

$$C \precsim_a C_a \precsim_a C' \quad \text{and} \quad C' \precsim_b C_b \precsim_b C,$$

and preferences between coalitions other than C_a, C_b are unchanged. Notice that at least one above preference of b is strict. Assume that

$$C' \prec_b C_b;$$

the argument in the other case is symmetric. Since C, C_b, C_a cannot form a prohibited 3-cycle, we have

$$C_b \precsim_c C_a.$$

Then, however, the coalitions C', C_b, C_a form a prohibited 3-cycle—a contradiction that proves the claim. *Q.E.D.*

LEMMA 8: *Let* $\mathcal{C}$ *satisfy* C1 *and* C3, *and let* **R** *satisfy* R1. *If no profile in* **R** *admits a* 3-*cycle prohibited by Lemma* 6, *then all profiles in* **R** *are pairwise aligned*.

PROOF: Take any proper coalitions C, C' such that $C \precsim_a C'$. We are to show the following claim: $C \precsim_b C'$ for any $b \in C \cap C'$. In particular, we may assume that $b \neq a$ and $C' \precsim_b C$.

Step 1. We prove the claim assuming that $a, b \in W$ and $|F| \leq 1$. Consider two cases:

(i) *Case* $W \nsubseteq C \cup C'$: Take $c \in W - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ and $C_b = \{b, c\}$ are coalitions. The assumptions of Lemma 7 are satisfied and our claim follows.

(ii) *Case* $W \subseteq C \cup C'$: By C1 and C3(i), $\{a, b\}$ is a coalition. Because of R1, we can assume that

$$C \precsim_a \{a, b\} \precsim_a C'.$$

By C3(ii), $W \nsubseteq C \cup \{a, b\}$, and thus the first preference above and case (i) imply that $C \precsim_b \{a, b\}$. Similarly, $\{a, b\} \precsim_b C'$. By transitivity, $C \precsim_b C'$.

Step 2. We prove the claim assuming that a or b is in F. The argument resembles Step 1.

(i) *Case* $W \nsubseteq C \cup C'$: Assumption C1 implies that one of the agents a, b is a worker. Assume that $a \in F$, $b \in W$; the argument in the other case is similar. Take $c \in W - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ is a coalition and either $C_b = \{b, c\}$ is a coalition or there is $f \in F - \{a\}$ such that $C_b = \{b, c, f\}$ is a coalition. In both cases, the assumptions of Lemma 7 are satisfied and our claim follows.

(ii) *Case* $W \subseteq C \cup C'$: The argument follows the argument of Step 1(ii) word by word.

Step 3. We prove the claim assuming that $a, b \in W$ and $|F| \geq 2$. Consider two cases.

(i) *Case* $F \nsubseteq C \cup C'$: Take $c \in F - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ and $C_b = \{b, c\}$ are coalitions, and our claim follows from Lemma 7.

(ii) *Case* $F \subseteq C \cup C'$: By C1, there is $c \in F - C'$, and then C1 and C3(i) imply that $\{a, c\}$ and $\{b, c\}$ are coalitions. If $c \notin C$, then Lemma 7 concludes the argument. Consider $c \in C$. By R1, we can assume that

$$C \precsim_a \{a, c\} \precsim_a C' \quad \text{and} \quad C' \precsim_b \{b, c\} \precsim_b C.$$

Step 2 applied to $\{b, c\} \precsim_b C$ gives $\{b, c\} \precsim_c C$ and we derive $C \precsim_c \{a, c\}$ similarly. By transitivity,

$$\{b, c\} \precsim_c \{a, c\}.$$

Since we also know that $\{a, c\} \precsim_a C' \precsim_b \{b, c\}$, the lack of prohibited 3-cycles gives

$$C' \sim_a \{a, c\} \sim_c \{b, c\} \sim_b C'.$$

Putting together what we have shown above about the preferences of c, we see that $C \precsim_c \{a, c\} \sim_c \{b, c\} \precsim_c C$ and, thus, $\{b, c\} \sim_c C$. Step 2 then implies that $\{b, c\} \sim_b C$. This indifference and the above-displayed indifference of b imply that $C \sim_b C'$. This ends the proof of the lemma because Steps 1–3 cover all possible situations. *Q.E.D.*

PROOF OF THEOREM 2: The proof of Theorem 2 follows from Lemmas 6 and 8. *Q.E.D.*

PROOF OF PROPOSITION 1: The assumptions on U_a guarantee that the maximization problem $\max_{(s_a)_{a\in C}} \log(\prod_{a\in C} U_a(s_a))$ is concave and has an interior solution. The resulting sharing rule is pairwise aligned and efficient. The remaining implication is the nontrivial part of the proposition. Let us thus assume that a sharing rule D is pairwise aligned and efficient, and construct the Nash-like representation. We may also assume that there is at least one worker, because otherwise C1 would imply that only singleton sets are coalitions and the efficiency of D would imply that any profile of U_a represents D, and thus the proposition would hold true.

For each proper coalition C and agents $a, b \in C$, let us define the function $t_{b,a} : R_+ \to R_+$ by

$$t_{b,a}(D_{a,C}(y)) = D_{b,C}(y), \quad y \geq 0.$$

The function maps the share agent a obtains when C produces y into the share agent b obtains at the same output level. The function is well defined because $D_{a,C}$ is onto R_+ and the strict monotonicity of $D_{a,C}$ guarantees that $D_{a,C}(y) = D_{a,C}(y')$ implies $y = y'$ and, hence, $D_{b,C}(y) = D_{b,C}(y')$. Moreover, function $t_{b,a}$ is strictly increasing and continuous (by monotonicity and continuity of $D_{a,C}$ and $D_{b,C}$). Finally, function $t_{b,a}$ does not depend on C because the pairwise alignment of D implies that for $C, C' \ni a, b$, the equality $D_{a,C}(y) = D_{a,C'}(y')$ is equivalent to $D_{b,C}(y) = D_{b,C'}(y')$.

To construct functions U_a, $a \in A$, fix an arbitrary reference worker w^*. By C1, functions $t_{w^*,b}$ are defined for all agents b. Furthermore, each function $t_{w^*,b}$ is invertible, the function $f : (0, \infty) \to (0, \infty)$ given by

$$f(t) = \min_{b\in A} [(t_{w^*,b})^{-1}(t)]^{-1/2}, \quad t > 0,$$

is continuous and strictly decreasing (because $t_{w^*,b}$ are continuous and strictly increasing), and $f(s) \to +\infty$ as $s \to 0+$ (because the inverse function $t^{-1}_{w^*,b}(s) \to 0$ as $s \to 0$). Thus, the auxiliary functions $\psi_a : (0, \infty) \to (0, \infty)$ given by

$$\psi_a = f \circ t_{w^*,a}$$

are positive, continuous, and strictly decreasing, and $\psi_a(s) \to +\infty$ as $s \to 0+$. Functions ψ_a are integrable at 0 because they are positive and bounded above by $(0, \infty) \ni s \to s^{-1/2}$. Define

$$W_a(s) = \int_0^s \psi_a(\tau)\, d\tau$$

and observe that W_a are strictly increasing, strictly concave, and differentiable, and $W_a'(0) = \lim_{s\to 0} \psi_a(s) = +\infty$. Define $U_a = \exp \circ W_a$ and notice that these functions are strictly increasing, strictly log-concave, and differentiable, and $\frac{U_a}{U_a'}(0) = \frac{1}{W_a'}(0) = 0$.

It remains to take an arbitrary coalition C and show that $(D_{a,C}(y))_{a\in C}$ is equal to the solution of the maximization problem

$$\arg \max_{\sum_{a\in C} s_a \le y} \sum_{a\in C} W_a(s_a).$$

The maximization problem is concave and hence has a solution. Furthermore, $W_a'(0) = +\infty$ guarantees that the solution is internal and satisfies the first order Lagrange condition $\psi_a(\tilde{s}_a) = \lambda$ for some constant λ. The first order condition can be rewritten as $t_{w^*,a}(\tilde{s}_a) = f^{-1}(\lambda)$ or

$$\tilde{s}_a = t_{a,w^*}(f^{-1}(\lambda)).$$

Since t_{a,w^*} is strictly monotonic, $\tilde{s}_a$ is uniquely determined by this equation and the tight feasibility constraint

$$\sum_{a\in C} \tilde{s}_a = y$$

(the constraint is tight because W_a are strictly increasing).

If C does not contain any workers, then C1 implies that C is a singleton coalition, and the Pareto efficiency of D is enough to yield the claim. Otherwise, fix a worker $w \in C$ and notice that for agents $a \in C$, we have $D_{a,C} = t_{a,w} \circ D_{w,C}$. Since C1, C2, and R1 are satisfied, Lemma 3 implies that $t_{a,w^*} \circ t_{w^*,w} = t_{a,w}$ and, hence,

$$D_{a,C}(y) = t_{a,w^*}(t_{w^*,w} \circ D_{w,C}(y)) = t_{a,w^*}(x)$$

for $x = t_{w^*,w} \circ D_{w,C}(y)$. Notice that x does not depend on a and can be written as $f^{-1}(\lambda)$ for some λ; thus, the system of equations for $D_{a,C}(y)$ is identical to the analogous equations for $\tilde{s}_a$ above. Also note that the Pareto efficiency of $D_{a,C}(y)|_{a\in C}$ is identical to the tight feasibility constraint on $\tilde{s}_a$. The uniqueness of solutions to these equations means that $\tilde{s}_a = D_{a,C}(y)$, which was to be proved. *Q.E.D.*

PROOF OF PROPOSITION 3: The representation of Corollary 2 is applicable because $\mathcal{C}$ satisfies C1–C3. Hence all regular, efficient, and stability-inducing sharing rules can be represented by a profile of agents' bargaining functions, U_a. Let d_a be the inverse function of $\frac{U_a'}{U_a}$. Let us refer to functions d_a as *demand* functions because $d_a(p)$ can be interpreted as the demand of an agent with utility $\log U_a$ who faces price p per unit of output. Notice that agent a's share s_a of output y in coalition $C \ni a$ satisfies $s_a = d_a(p)$, where p is determined by the market-clearing condition $\sum_{a\in C} d_a(p) = y$. Agents prefer coalitions with lower market-clearing price p, and the sharing rule is fully characterized by the profile of demands $(d_a)_{a\in A}$. For instance, if $U_a(s) = s^{\lambda_a}$, then $d_a(p) = \frac{\lambda_a}{p}$ and the market clearing price in coalition of such agents is $p = \frac{1}{y}\sum_{a\in C}\lambda_a$.

Notice that every demand function $d_a : R_+ \to R_+$ is a decreasing bijection, and any decreasing bijection d_a corresponds to a bargaining function $U_a(s) = \exp \circ \int_1^s d_a^{-1}(t)\,dt$ that is strictly increasing, differentiable, log-concave, and such that $\frac{U_a}{U_a'}(0) = 0$. Consequently, two demand profiles $(d_a)_{a\in A}$ and $(\tilde{d}_a)_{a\in A}$ represent the same sharing rule if and only if there is an increasing bijection $T : R_+ \to R_+$ such that $\tilde{d}_a = d_a \circ T$ for every $a \in A$. We can thus normalize the demand function by assuming that $\sum_{a\in A} d_a(p) = \frac{1}{p}$. In this way, we obtain a one-to-one correspondence between sharing rules and normal-form demand profiles. Let us partition the sharing rules into subclasses such that normal-form demands d_b, $b \in F \cup (W - L)$, and the sum $\sum_{a\in L} d_a$ are the same for each rule in the subclass. It remains to prove that among rules in an element of the partition, the unique rule in which $d_w = d_{w'}$ for workers $w, w' \in L$ maximizes the probability that L belongs to a stable coalition structure.

We can restrict attention to output profiles that result in strict preferences of agents among subcoalitions of $A - L$, as this event happens with probability 1. Furthermore, we can prove the claim conditional on a fixed profile of outputs for L and coalitions of agents in $A - L$. Because of strict preferences and the regularity of the sharing rule, Theorem 1 implies that there is a unique stable coalition structure on $A - L$. Let $\{C^1, \dots, C^k\}$ be this stable coalition structure. Hence, whenever L belongs to a stable coalition structure, the stable structure is $\{L, C^1, \dots, C^k\}$. We refer to this event as L being stable.

Let us take any sharing rule and let $(d_a)_{a\in A}$ be its normal-form representation. Select a pair of workers $w, w' \in L \cap W$ and adjust the sharing rule so that both workers w and w' are endowed with demand functions $\frac{d_w + d_{w'}}{2}$. The resulting demand-function representation is in normal form and the resultant sharing rule belongs to the same element of the partition as $(d_a)_{a\in A}$. Let us check that the probability of L being stable is larger for the adjusted sharing rule (and strictly larger if the distributions are strictly log-concave). The independence of output distributions implies that the probability that L is stable equals the product of probabilities that L is not blocked by coalitions $C \neq L$ such that $C \cap L \neq \emptyset$. The adjustment does not change the probability that L is blocked by C if C contains both or neither of workers w and w' because the market-clearing prices of L, coalitions disjoint with L, and C stay the same in every profile of outputs. We show that the adjustment increases the joint probability that L is not blocked by coalitions C_w or $C_{w'}$ such that $w \in C_w$ and $w' \notin C_w$, and $C_{w'} = C_w \cup \{w'\} - \{w\}$; by symmetry, either both sets C_w and $C_{w'}$ are coalitions or none is. Let us denote the set $C_w - \{w\} = C_{w'} - \{w'\}$ by S (this set is not necessarily a coalition), and let p^* be the minimum among market-clearing prices in coalition L and those among coalitions C^i, $i = 1, \dots, k$, that have a nonempty intersection with set S. Coalitions C_a and C_b do not block $\{L, C^1, \dots, C^k\}$ precisely when their market clearing prices are above p^*. Denote by F_m the c.d.f. of the probability distribution from which outputs of a coalition of size m are drawn. The probability that L is not blocked by C_w and $C_{w'}$ before adjustment equals

$$F_{|C_w|}\left(y \le d_w(p^*) + \sum_{a\in S} d_a(p^*)\right) F_{|C_{w'}|}\left(y \le d_{w'}(p^*) + \sum_{a\in S} d_a(p^*)\right),$$

and by concavity of $\log \circ F_{|C_w|} = \log \circ F_{|C_{w'}|}$, the probability after adjustment

$$F_{|C_w|}\left(y \le \frac{d_w + d_{w'}}{2}(p^*) + \sum_{a\in S} d_a(p^*)\right) \times F_{|C_{w'}|}\left(y \le \frac{d_w + d_{w'}}{2}(p^*) + \sum_{a\in S} d_a(p^*)\right)$$

is higher.

As we repeat the above procedure in such a way that every possible pair of workers a, b is selected infinitely many times, the demand function of each worker from L converges pointwise to $\frac{1}{|L\cap W|}\sum_{w\in L\cap W} d_w$ (the demand functions of other agents do not change). The probability that L is stable is higher for the sharing rule that is the pointwise limit of the procedure than for the original sharing rule $(d_a)_{a\in A}$. This is so because the distributions of outputs are absolutely continuous, demands are decreasing, and, hence, the probability of L being stable is continuous in the point-wise metric on the demand profiles $(d_a)_{a\in A}$. *Q.E.D.*

PROOF OF PROPOSITION 4: First consider the case $L = A$. There are two places in the sketch of the proof in the main text that require a supporting argument. First consider coalition C that contains a but not b. We need to show that the probability that A is not blocked by one (or both) of the coalitions $C_a = C$ and $C_b = C \cup \{b\} - \{a\}$ is decreasing when we replace λ_a with $\lambda_a + \varepsilon$ and λ_b with $\lambda_b - \varepsilon$ so that

$$\lambda_a < \lambda_a + \varepsilon \le \lambda_b - \varepsilon < \lambda_b.$$

We refer to such changes of bargaining powers as a $(\lambda_a, \lambda_b, \varepsilon)$ adjustment. We show that $(\lambda_a, \lambda_b, \varepsilon)$ adjustment increases the probability that A is not blocked conditional on the output of A being equal to an arbitrary $y \ge 0$. The argument follows the same logic as an analogous argument in Proposition 3. Denote by F_k the c.d.f. of the probability distribution from which outputs of a coalition of size k are drawn. Conditional on the output of A being equal to y and keeping the original bargaining powers, the grand coalition is not blocked by either C_a and C_b if

$$\frac{y}{\sum_{i\in A}\lambda_i} \geq \frac{\mathbf{y}(C_a)}{\sum_{i\in C_a}\lambda_i} \quad \text{and} \quad \frac{y}{\sum_{i\in A}\lambda_i} \geq \frac{\mathbf{y}(C_b)}{\sum_{i\in C_b}\lambda_i}.$$

By independence, the conditional probability of these two inequalities is equal to the product

$$F_{|C_a|}\left(\frac{\sum_{i\in C_a}\lambda_i}{\sum_{i\in A}\lambda_i}y\right)F_{|C_b|}\left(\frac{\sum_{i\in C_b}\lambda_i}{\sum_{i\in A}\lambda_i}y\right).$$

Similarly, the conditional probability that A is not blocked after we adjust λ_a and λ_b equals

$$F_{|C_a|}\left(\frac{\varepsilon+\sum_{i\in C_a}\lambda_i}{\sum_{i\in A}\lambda_i}v\right)F_{|C_b|}\left(\frac{-\varepsilon+\sum_{i\in C_b}\lambda_i}{\sum_{i\in A}\lambda_i}v\right),$$

and is higher than the probability before adjustment because of the assumption that $\log\circ F_{|C_a|}=\log\circ F_{|C_b|}$ is concave. The probability increase is strictly positive if $\log\circ F_{|C_a|}=\log\circ F_{|C_b|}$ is strictly concave.

The remaining supporting argument is given by the following claim.

CLAIM: *If* $\sum_{a\in A}\lambda_a=\sum_{a\in A}\lambda'_a$ *and* $(\lambda_a)_{a\in A}>(\lambda'_a)_{a\in A}$, *then there exists a finite sequence of* $(\lambda_a,\lambda_b,\varepsilon)$ *adjustments that transforms* $(\lambda'_a)_{a\in A}$ *into* $(\lambda_a)_{a\in A}$ (*that is, there are bargaining power profiles* $\lambda^k\in R_+^A$, $k=1,\dots,n\geq 2$, *such that* $\lambda^1=\lambda'$, $\lambda^n=\lambda$, *and there are two coordinates* $a,b\in C$ *such that* $\lambda^{k+1}_{-a,b}=\lambda^k_{-a,b}$, *and* $\lambda^k_a<\lambda^{k+1}_a\leq\lambda^{k+1}_b<\lambda^k_b$, *and* $\lambda^{k+1}_a-\lambda^k_a=\lambda^k_b-\lambda^{k+1}_b$).

PROOF: We may assume that $\lambda'_1<\cdots<\lambda'_n$ because the ordering between λ' and λ is independent of permutation (or renaming) of agents in A. Then also $\lambda_1<\cdots<\lambda_n$ because $\lambda>\lambda'$. Notice that $\lambda_1>\lambda'_1$, as otherwise $\lambda_i\leq\lambda'_i$ for all $i=1,\dots,n$ with some inequalities strict, contrary to λ and λ' having the same sum of coordinates. Similarly, $\lambda_n<\lambda'_n$.

Let $\lambda^1=\lambda'$ and define a to be the maximal subscript such that $\lambda^1_a=\lambda^1_1$, and define b to be the minimal subscript such that $\lambda^1_b=\lambda^1_n$. Let $\varepsilon=\min(\lambda^1_b-\lambda^1_{b-1},\lambda^1_b-\lambda_1,\lambda^1_{a+1}-\lambda^1_a,\lambda_1-\lambda^1_a)$. Let λ^2 be given by the $(\lambda^1_a,\lambda^1_b,\varepsilon)$ adjustment of λ^1 (in words, the adjustment is raising the a-coordinate and lowering the b-

coordinate as long as λ_a^2 is weakly lower than λ_{a+1}^1 and λ_n, and λ_b^2 weakly higher than $\lambda_{a+1}^1$and λ_1). Notice that

$$\lambda^1 < \lambda^2 \leq \lambda$$

and

$$\sum_{i=1,\dots,n} |\lambda_i - \lambda_i^1| > \sum_{i=1,\dots,n} |\lambda_i - \lambda_i^2|.$$

If $\lambda^2 \neq \lambda$, then we construct λ^3 via the same procedure with λ^2 substituted for λ^1. The analogs of the above-displayed relations continue to hold. Since all λ_a^k and λ_b^k come from a finite grid, the iterations terminate with some $\lambda^k = \lambda$. This ends the proof of the claim. *Q.E.D.*

Finally, consider the case $L \neq A$. Again, we may assume that $\sum_{a\in L} \lambda_a = \sum_{a\in L} \lambda'_a$. The argument that ε adjustments increase the probability of L being stable follows the same lines as an analogous argument in the proof of Proposition 3. The analog of the above claim on sequences of ε adjustments remains true, and concludes the proof.[28] *Q.E.D.*

PROOF OF PROPOSITION 5: Fix agents' preference profiles and a stable coalition structure $\{C_1, \dots, C_k\}$. To show that the coalition structure is implementable as a strong Nash equilibrium, consider first the case $C_1 = A$. By assumptions of the proposition, there is a profile of strategies of agents σ such that $A \in \mu(\sigma)$; this profile must be a strong Nash equilibrium. If $C_1 \neq A$, then all coalitions C_i are proper. By Lemmas 3 and 4, there are no n-cycles for $n = 2, 3, \dots$. Notice that this implies that at least one of the coalitions $C_1, \dots, C_k$ is weakly preferred by its members to all other proper coalitions. Indeed, if not, then let C'_1 be a proper coalition that is strictly preferred to C_1 by an agent a_1 from $C_1 \cap C'_1$. Since $\{C_1, \dots, C_k\}$ is stable, there must be a coalition C_i and agent a'_1 that weakly prefers C_i to C'_1. We could repeat the procedure and define C'_2 to be a proper coalition that is strictly preferred to C_i by at least one agent $a_2 \in C_i \cap C'_2$. Since there are a finite number of coalitions, in this way we would eventually construct an n-cycle. The contradiction proves that there is a coalition C_{i_1} that is weakly preferred by all its members to any

[28]We can define a similar partial ordering on all regular, efficient, and stability-inducing sharing rules. Take, for instance, $L = A$ and represent the sharing rule in the demand form of the proof of Proposition 3. The ordering is then such that $(d_a)_{a\in A}$ dominates $(d'_a)_{a\in A}$ if $\frac{d_{(i)}(p)}{d_{(i+1)}(p)} \geq \frac{d'_{(i)}(p)}{d'_{(i+1)}(p)}$, $i = 1, \dots, |A|$, where the order statistics $d_{(i)}$ and $d'_{(i)}$ are taken pointwise (independently for every price p). The comparison of probabilities of A being stable under the two sharing rules follows the construction from Proposition 4 with the proviso that (d_a, d_b, ε) adjustments be defined for functions $\varepsilon : R_+ \to R_+$ that preserve monotonicity of $d_a + \varepsilon$ and $d_b - \varepsilon$.

other proper coalition. We can recursively re-index coalitions $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ so that C_{i_j} is weakly preferred by all its members to any proper coalition of agents in $A - (C_{i_1} \cup \cdots \cup C_{i_{j-1}})$. By assumptions of the proposition, there is a profile σ_{C_i} of strategies of agents from C_i that leads to the formation of C_i. Profiles σ_{C_i} put together form a strong Nash equilibrium. *Q.E.D.*

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Manuscript received May, 2007; final revision received June, 2011.