# Stochastic vs Deterministic Mechanisms in Multidimensional Screening

Marek Pycia[1]
Massachusetts Institute of Technology

May 20, 2006

Abstract

This paper studies the multidimensional screening problem of a profit-maximizing monopolist who designs and sells goods with multiple indivisible attributes. The buyer's utility is linear in the probabilities of obtaining the attributes. The values of the attributes are buyer's private information. The paper solves the seller's problem for an arbitrary number of attributes when there are two types of buyers. When there is a continuum of buyer types, the paper shows that generically the seller wants to sell goods with some of the attributes partly damaged, stochastic, or leased on restrictive terms. In particular, the often-studied simple bundling strategies are shown to be generically suboptimal. This last result is qualified in the case of two buyer types. In this case, the maximum seller's loss from the restriction to simple bundling is 12.5%.

[1]Massachusetts Institute of Technology, http://alum.mit.edu/www/marek.pycia This is the third chapter of my PhD dissertation at MIT. Two-dimensional version of Sections 2-4 constituted the second chapter of my M.Phil (DEEQA) project at University of Toulouse 1, Pycia (2000). I would like to thank Jean-Charles Rochet for his advice and calling my attention to the problem, and Marco Ottaviani for his many comments. I would also like to thank Abhijit Banerjee, Glenn Ellison, Robert Gibbons, Bengt Holmström, Sergei Izmalkov, David McAdams, Anna Myjak-Pycia, Jean Tirole, and seminar participants at Toulouse, MIT, and the Stony Brook Game Theory Festival. This project was supported in part by the Leon Koźmiński Academy of Entrepreneurship and Management in Warsaw.

## 1. Introduction

Determining the optimal design of a product line of goods with multiple attributes when a monopolistic firm sells to buyers with unknown valuations is a long-standing unsolved problem.[2]Among the strategies to approach it, McAfee and McMillan's (1988) proposal has proved particularly influential.[3] McAfee and McMillan (1988) consider a monopolist who designs and sells a product line of goods with several indivisible attributes.[4] The buyers' utility is linear in price and in the probabilities of obtaining the attributes.[5] The values of the attributes are buyers' private information. The monopolist has zero marginal cost and aims to maximize the expected revenue subject to buyers' incentive and participation constraints. McAfee and McMillan argued that the problem may be reduced to finding the optimal menu of deterministic, or simple, bundles of attributes. In effect, the subsequent literature[6] focused on finding the optimal deterministic bundles; the corresponding class of seller's strategies has been referred to as simple bundling.[7]

McAfee and McMillan's claim is known to be true in the case of one attribute solved by Riley and Zeckhauser (1983).[8] One way to understand the intuition behind the one-dimensional case is to think of the problem as a single unit auction with one buyer, in which setting a reservation price is an optimal strategy (Myerson (1983), Bulow and Roberts (1989)).[9] Recently, however, several authors including Pycia (2000), Manelli

[2]A good with multiple attributes is equivalent to a bundle of commodities. Wilson (1993), Armstrong (1996), Rochet and Choné (1998), Armstrong (1999), Armstrong and Rochet (1999), Thanassoulis (2004), and Manelli and Vincent (2004) study special cases of the problem. Rochet and Stole (2003) survey the literature.

[3]Cf. Avery and Hendershott (2000), Miller, Piankov, and Zeckhauser (2001), and papers cited in footnote 6.

[4]This model is studied in the third part of McAfee and McMillan (1988).

[5]Stochastic presence of an attribute may be interpreted as lower quality, limited quantity, restrictive lease terms, or damaging a la Deneckere and McAfee (1996).

[6]Cf. McAfee, McMillan, and Whinston (1989), McAdams (1998), and Manelli and Vincent (2006).

[7]The simple bundling strategies include both pure and mixed bundling of Adams and Yellen (1976).

[8]Similar one-dimensional results were proved by Stokey (1979), Fudenberg and Tirole (1983), and Courty and Hao (2000). A more general one-attribute problem was solved by Mussa and Rosen (1978).

[9]A more direct intuition for this result is as follows. Think of the seller as selling the attribute in probability increments. The incentive constraints imply that if the seller sells an increment to one buyer at a given price, then the seller needs to offer this probability increment to all higher valuation buyers at this price, and may charge a higher price only on additional increments. However, if the seller

and Vincent (2004), and Thanassoulis (2004) independently constructed counterexamples to show that there are distributions of agents' valuations for which the simple bundling strategies are suboptimal.[10]

This paper makes positive and negative contributions to the understanding of multidimensional screening in the setting proposed by McAfee and McMillan. On the positive side, the paper solves the problem when there are two buyer types and an arbitrary number of attributes. On the negative side, the paper proves that simple bundling strategies are generically suboptimal, and presents some estimates of the loss inherent in the restriction to simple bundling.

The positive contribution is developed in Section 3. As in the analysis of one-dimensional situations with two types, it is natural to refer to the buyer with the larger sum of values of all attributes as a high type, and to the other type of buyer as a low type. In an equilibrium, the high type buys the good with all attributes, and the low type buys the good with the attributes for which the ratio of low-type to high-type value is high enough. When the low type values at least one attribute more than the high type does, then the seller cannot post per-attribute prices, i.e., genuinely needs to bundle the attributes. When the high type values each attribute more than the low type does, the problem may be viewed as a collection of one-dimensional subproblems. Profit maximization generically requires randomization – that is simple bundling is suboptimal – except when the high type obtains an informational rent and when the low type is excluded from the market. As in one-dimensional problems, the high type obtains an informational rent when the high type is relatively scarce. The low type never has a rent and is excluded from the market when the low type is relatively scarce.

To introduce the negative contribution of the paper let us look at an example in which the distribution of buyer types require the seller to employ more complex strategies than simple bundling of attributes. Consider a software company that serves a 50%-50% population of professional and occasional users, and sells software packages with two potential attributes: reliability and ease of use. Assume that the professional users are

weakly prefers to sell the first increment at a lower rather than higher price, than the seller weakly prefers to sell all subsequent increments at the lower price.

[10] Similarly, the simple unidimensional intuition is not robust when the outside options are type dependent. Deneckere and McAfee (1996), Rasul and Sonderegger (2001), and Ambjørnsen (2002), and Figueroa and Skreta (2005) find that in some cases type dependent outside options lead to stochastic screening (interpretation favoured by Rasul and Sonderegger (2001) and Figueroa and Skreta (2005)) or to damaged goods (Ambjørnsen (2002), Deneckere and McAfee(1996)).

willing to pay \$300 for reliability and nothing for the ease of use, while the occasional users are willing to pay \$100 for reliability and also \$100 for the ease of use. If restricted to simple (deterministic) bundles, the best seller's choices are:

- sell reliable software to professional users for the price of \$300 and easy to use software to occasional users for \$100, or
- sell the reliable and easy to use software to both types of users for \$200.

Both choices lead to the expected revenue and profits of \$200 per user. Using stochastic mechanisms, the seller can achieve higher profits. The seller can sell reliable software to professional users for \$300 and easy to use but only 50% reliable[11] software to occasional users for \$150. This menu of contracts is incentive compatible and leads to the expected profit of \$225 per buyer, and hence is 12.5% better than the best simple bundling strategy.

The negative contribution of the paper is developed in Sections 4 and 5. Section 4 shows that generically — that is on an open and dense set of Lebesgue absolutely continuous distributions — simple bundling strategies are suboptimal in the McAfee and McMillan model. Section 5 qualifies this result by showing that if there are two buyer types, then the maximum seller's loss from not being able to use lotteries is 12.5%, as in the example discussed above.

## 2. Model

The formal model is the same as that studied by McAfee and McMillan (1988). A monopolistic seller sells a good with $n$ indivisible attributes to a buyer who desires at most one unit of each attribute. The utility[12] of buyer of type $t \in [0,1]^n$ from a contract

$$(q;p) \in [0,1]^n \times [0,\infty)$$

composed of price $p$ and the vector of probabilities $q_i$ of receiving attributes $i = 1, ..., n$ is given by

$$U(q,p;t) = tq - p = t_1 q_1 + ... + t_n q_n - p.$$

[11] Cf. footnote 5.

[12] We adapt the standard assumption that whenever the buyer is indifferent he chooses a contract that brings most profit to the seller.

The buyer's reservation value equals 0. We denote by $F$ the seller's prior distribution over the buyer's types. The seller's valuation or production cost of the goods is normalized to be zero, and she seeks to maximize her expected revenue

$$\max \int p(t)\, dF(t) \tag{1}$$

subject to the feasibility, participation, and incentive-compatibility constraints of the buyer

$$q(t) \in [0,1]^n,$$

$$tq(t) - p(t) \geq 0 \text{ for all } t \in \text{supp}(F), \tag{IR}$$

$$tq(t) - p(t) \geq tq(t') - p(t') \text{ for all } t, t' \in \text{supp}(F). \tag{IC}$$

A product $q \in [0,1]^n$ and contract $(q; p)$ offered by the monopoly is called *simple* (or deterministic) if $q \in \{0,1\}^n$. A contract is called *complex* (or stochastic) if $q \notin \{0,1\}^n$. A product line is called simple if it contains only simple products. Otherwise, the product line is called complex. Similarly, a menu of contracts is called simple if it contains only simple contracts, and called complex otherwise.

## 3. Two Buyer Types and an Arbitrary Number of Attributes

This section solves (1) in the case of two buyer types and an arbitrary number of attributes. The seller faces

- a buyer of type $A = (a_1, ..., a_n) \in [0,1]^n$ with probability $\mu_A$, and
- a buyer of type $B = (b_1, ..., b_n) \in [0,1]^n$ with probability $\mu_B = 1 - \mu_A$.

For simplicity of exposition, assume that each attribute is positively valued by at least one buyer, that is

$$\max\{a_i, b_i\} > 0 \text{ for } i = 1, ..., n. \tag{2}$$

Assume also that $A$ values the contract offering all attributes weakly more than $B$ does, that is

$$a_1 + ... + a_n \geq b_1 + ... + b_n. \tag{3}$$

We may thus think of $A$ as the high type and of $B$ as the low type. Finally, let us also reindex the attributes so that

$$\frac{b_i}{a_i} \text{ is a weakly increasing sequence.} \tag{4}$$

All these assumptions are without loss of generality.

We will show that in an optimal product line, whenever a product contains attribute $i$, then it contains all attributes $j > i$. The high type will buy a product with all attributes, and the low type will buy a product with all attributes above some cut-off level. The cut-off level will be shown to be the lower of two potential cut-offs[13]

$$n^* = \min\left\{i : a_{i+1} + ... + a_n < b_{i+1} + ... + b_n\right\}, \tag{5}$$

and

$$n^{**} = \min\left\{i : \frac{b_i}{a_i} \geq \mu_A\right\}.$$

In the low type aimed product, the cut-off attribute $n^*$ may be randomized and offered with probability

$$\pi = \frac{b_{n^*+1} + ... + b_n - a_{n^*+1} - ... - a_n}{a_{n^*} - b_{n^*}}. \tag{6}$$

Notice that $\pi$ is well defined and belongs to $(0, 1]$ if $n^* < +\infty$.

Using the above introduced notation, the following theorem gives a full characterization of the two-type case for an arbitrary number of attributes $n = 1, 2, ....$[14]

**Theorem 3.1.** Assume (2), (3), and (4).
If $b_n \leq a_n$ or $n^{**} \leq n^*$, then the following simple menu of contracts is optimal[15]

- $(1, ..., 1; a_1 + ... + a_{n^{**}-1} + b_{n^{**}} + ... + b_n)$,
- $\left(\underbrace{0, ..., 0}_{n^{**}-1}, \underbrace{1, ..., 1}_{n-n^{**}+1}; b_{n^{**}} + ... + b_n\right)$.

If $a_n < b_n$ and $n^* < n^{**}$, then the following menu of contracts is optimal

- $(1, ..., 1; a_1 + ... + a_n)$,

[13] By convention, $\min \emptyset = +\infty$.

[14] The result may be generalized to the case with continuum of attributes.

[15] If $n^{**} = \min \emptyset = +\infty$, then the menu is reduced to offering $(1, ..., 1; a_1 + ... + a_n)$ to $A$ and shutting $B$ out of the market.

- $\left(\underbrace{0,...,0}_{n^*-1},\pi,\underbrace{1,...,1}_{n-n^*};\pi b_{n^*}+b_{n^*+1}+...+b_n\right).$

In the latter case, generically $\pi\in(0,1)$ and the simple menus are suboptimal.

The basic intuition for the theorem relies on the fact that — except for the case when both buyer types buy the good $(1,...,1)$ — the IC constraint of type $B$ is slack while the IR constraint of type $B$ and the IC constraint of type $A$ are tight. The first case corresponds to tight IR constraint of type $A$ and second case corresponds to this last constraint being slack. The formal proof is in the appendix.

Let us finish with several remarks.

(1) As in Section 3, if $A$ and $B$ derive same the value from the bundle of all attributes then the optimal contract is simple and offers a single good with all attributes for the common valuation. In terms of Theorem 4.1, the equality of valuations means that $n^*=1$ and $\pi=1$. The optimal menu falls thus under the first case if $n^{**}=1$ and under the second case otherwise.

(2) If the high type has weakly higher valuation for all attributes, then the seller may allow the buyers to compose the good from separately priced attributes. As in Proposition 3.1, the optimal contract is simple and offers attributes $i=1,...,n$ at price $b_i$ if $b_i\geq a_i\mu_A$ and at price $a_i$ if $b_i<a_i\mu_A$.

If there are attributes that the low type values more than the high type, then the seller needs to bundle the attributes and cannot price them separately.

(3) In equilibrium, the high type buys a good with all attributes, while the low type buys the attributes $\min\{n^*,n^{**}\},...,n$. The attributes bought by the low type include all those that the low type values weakly more than the high type does, and may include some of the remaining attributes.

(4) There exists an optimal menu of contracts that includes a simple contract and a contract that randomizes over at most one of the attributes.

(5) Profit maximization generically requires complex menus except if $b_n\leq a_n$ or $n^{**}\leq n^*$. This last condition is equivalent to $\mu_A\leq\frac{b_{n^*}}{a_{n^*}}$. Thus, complex menus are called for if the problem is genuinely multidimensional $(b_n>a_n)$ and there are enough high types in the population $(\mu_A\leq\frac{b_{n^*}}{a_{n^*}})$.

(6) Generically, the high type obtains an informational rent if

- the high type is scarce in the sense $\mu_A \leq \frac{b_{n^*}}{a_{n^*}}$, or
- the problem is reducible to a collection of one-dimensional problems ($b_i \leq a_i$ for $i = 1, ..., n$) and the seller is not shutting the low type out of the market ($\mu_A a_n < b_n$).

Otherwise no type obtains a rent.

Finally notice that screening a continuous distribution of buyers close to the two-type distributions requiring complex product lines also requires complex product lines. This last point is developed in the next section.

## 4. The Generic Suboptimality of Simple Bundling

This section shows that the generic distribution of buyer types induces the seller to offer menus of complex contracts. “Generic” in this context means that the set of distributions that require the seller to use complex contracts in order to maximize profits contains a dense and open subset of the space of all distributions. The relevant space of distributions is the space of Lebesgue absolutely continuous Borel probability measures on $[0, 1]^n$ endowed with weak topology relative to bounded continuous functions.

The seller's problem (1) always has a solution. The seller's problem also has a solution if the seller is constrained to use simple bundling strategies.[16] The following result compares these two solutions.

**Theorem 4.1.** For a generic distribution $F$ of buyer types the monopolistic seller seeking to maximize (1) can earn strictly more by offering a menu of complex contracts than the maximum of expected earnings from menus of simple contracts, that is

$$\max_{q(t)\in[0,1]^n, IC, IR} \int p(t)\, dF(t) > \max_{q(t)\in\{0,1\}^n, IC, IR} \int p(t)\, dF(t)\,. \tag{7}$$

The proof is divided into two parts: density and openness. The proof of the density relies on the special structure of simple menus. The structure of simple menus allows

[16] Cf. Rochet and Choné's (1998) or Lemma 5.2.

us to locally perturb a distribution that does not satisfy (7) so that the resultant distribution satisfies (7). This perturbation is a mixture of the original distribution and a Lebesgue continuous approximation to a two-type distribution with complex solution (given by Proposition 3.2).[17] The proof of openness of the set of distributions satisfying (7) relies on Berge's maximum theorem and Rochet's (1985) reinterpretation of the IC conditions in terms of convexity of buyer's rent as a function of buyer's type. Let us start with the density proof and then discuss the framework used to prove openness.

*Proof of the density part of Theorem 5.1.* To show that the set of stochastic distributions is dense in the set of Lebesgue absolutely continuous probability distributions on $[0,1]^n$ , take a distribution $F$ such that a simple menu of contracts

$$u^* = \{(J; p^*_J) : J \in \{0,1\}^n\}$$

is optimal. Our goal is to construct a complex menu $u^h$ and a Lebesgue absolutely continuous distribution $G_\varepsilon$ such that the seller strictly prefers the menu $u^h$ to any simple menu if buyer types are distributed according to $(1-\alpha)F + \alpha G_\varepsilon$ and $\alpha > 0$ is small.

Notice that it is enough to consider the case when the density of $F$, denoted $f$, is continuous and its support contained in $[\delta, 1-\delta]^n$ for a small $\delta$. For any bundle $J \in \{0,1\}^n$, we may also assume the corresponding buyer type $t^J = J$ strictly prefers that the contract $(J; p^*_J)$ to all other contracts in $u^*$.[18]

To construct $u^h$, let us fix $h > 0$ and define an auxiliary menu of contracts

$$\tilde{u}^h = u^* - \left\{\left(1,0,...,0; p^*_{(1,0,0,...,0)}\right)\right\} \cup \left\{\left(1,0,...,0; p^*_{(1,0,0,...,0)} + h\right)\right\}.$$

Now,

$$u^h = \tilde{u}^h \cup \{(K; p^*_K)\},$$

[17]The genericity result of Theorem 4.1 is not limited to the space of Lebesgue absolutely continuous distributions with weak topology. An inspection of the proofs in this paper shows that the class of measures requiring complex contracts is dense in any space of distributions that contains the class of distributions with differentiable Lebesgue densities, and is endowed with topology satisfying the following three assumptions:

1) The class of distributions with differentiable densities is dense.

2) If $F_k \to F$ then for every continuous $u$ we have $\int u dF_k \to \int u dF$ (i.e. the topology is at least as strong as the weak topology).

3) $(1-\varepsilon)F + \varepsilon G \to F$ as $\varepsilon \to 0$.

Moreover, under these assumptions the proof in the paper shows that the class of distributions requiring complex contracts is locally open around each Lebesgue absolutely continuous distribution. Consequently, this class is generic in the space of Lebesgue absolutely continuous distributions.

[18]We refer to the preferences of types $t^J$ despite that they are not in the support of $F$.

where $K$ is the complex bundle $\left(1, \frac{1}{2}, 0, ..., 0\right)$ and $p^*_K = \frac{1}{2}p^*_{(1,0,...,0)} + \frac{1}{2}p^*_{(1,1,0,...,0)}$.

We want to show that profits from $u^h$ are the same as from $u^*$ up to first order in $h$ when buyer types are distributed according to $F$. For brevity, the profit that the seller obtains from menu $u^h$ if the buyer types are distributed according to $F$ will be referred to as the expected profit from $u^h$ over $F$. The expected profits from $\tilde{u}^h$ over $F$ equals that from $u^*$ over $F$ up to first order in $h$, because $u^*$ is optimal if buyer types are distributed according to $F$. Thus, it is enough to compare profits from $u^h$ and $\tilde{u}^h$.

For a menu of contracts $u$ and a complex contract $(J; p_J) \in u$, denote by $T^u_J$ the subset of buyer types that weakly prefer $(J; p_J)$ to other contracts in $u$. Notice that for any simple bundle $J \neq \{(1, 0, ..., 0)\}$, the types from $T^{u^*}_J$ weakly prefer $(J; p^*_J)$ to any other choice in $u^h$. Thus the difference in profits between $u^h$ and $\tilde{u}^h$ has to come from types in $T^{u^*}_{(1,0,0,...,0)} \cap T^{u^h}_K$. Furthermore, the mass of $T^{u^*}_{(1,0,0,...,0)} \cap T^{u^h}_K \cap T^{\tilde{u}^h}_J$ is of second order in $h$ if $J \neq (1, 0, ..., 0), (1, 1, 0, ..., 0)$. The impact on difference in profits in the two remaining subsets $T^{u^*}_{(1,0,0,...,0)} \cap T^{u^h}_K \cap T^{\tilde{u}^h}_{(1,0,0,...,0)}$ and $T^{u^*}_{(1,0,0,...,0)} \cap T^{u^h}_K \cap T^{\tilde{u}^h}_{(1,1,0,...,0)}$ cancel out because

- In the first subset $u^h$ brings $\frac{1}{2}\left(p^*_{(1,1,0,...,0)} - p^*_{(1,0,0,...,0)}\right)$ more per buyer than $\tilde{u}^h$ and the mass of this subset is up to first order
$$\left(E_{T^{u^*}_{(1,0,0,...,0)} \cap T^{u^*}_{(1,1,0,...,0)}} f\right)\left(\mathrm{vol}_{n-1} T^{u^*}_{(1,0,0,...,0)} \cap T^{u^*}_{(1,1,0,...,0)}\right) h.$$
- In the second subset $\tilde{u}^h$ brings $\frac{1}{2}\left(p^*_{(1,1,0,...,0)} - p^*_{(1,0,0,...,0)}\right)$ more per buyer than $u^h$ and the mass of this subset is up to first order
$$\left(E_{T^{u^*}_{(1,0,0,...,0)} \cap T^{u^*}_{(1,1,0,...,0)}} f\right)\left(\mathrm{vol}_{n-1} T^{u^*}_{(1,0,0,...,0)} \cap T^{u^*}_{(1,1,0,...,0)}\right) h.$$

Thus $u^h$ is first order equivalent to $\tilde{u}^h$ and hence to $u^*$.

To construct a distribution that is close to $F$ and requires complex contracts denote $R = \max\left\{0, p^*_{(1,1,0,...,0)} - p^*_{(1,0,0,...,0)} - p^*_{(0,1,0,...,0)}\right\}$ and consider a four-type auxiliary distribution $\tilde{G}$ with masses
$$\mu\left(t^1\right), \mu\left(t^2\right) >> \mu\left(t^3\right) >> \mu\left(t^4\right)$$
on points $t^i$ defined as follows:
$$\begin{aligned}
t^1 &= \left(p^*_{(1,1,0,...,0)} + h, 0, ..., 0\right), \\
t^2 &= \left(0, p^*_{(0,1,0,...,0)}, 0, ..., 0\right), \\
t^3 &= \left(p^*_{(1,0,0,...,0)} + R, p^*_{(1,1,0,...,0)} - p^*_{(1,0,0,...,0)}, 0, ..., 0\right), \\
t^4 &= t^3 + (h, -2h, 0, ..., 0).
\end{aligned}$$

By assumptions on $u^*$, the points $t^1, t^2, t^3, t^4 \in (0,1)^2 \times \{0\}^{n-2}$ for small $h$. In the spirit of Proposition 3.2, we can show that to extract maximum expected profit from $\tilde{G}$ seller may offer bundles $(1,0,0,...,0)\,,(0,1,0,...,0)\,,(1,1,0,...,0)\,,$ and $K$ at prices

$$\begin{aligned}
p_{(1,0,0,...,0)} &= t_1^1 = p^*_{(1,0,0,...,0)} + h, \\
p_{(0,1,0,...,0)} &= t_2^2 = p^*_{(0,1,0,...,0)}, \\
p_{(1,1,0,...,0)} &= t_1^3 + t_2^3 - R = p^*_{(1,1,0,...,0)}, \\
p_K &= t_1^4 + \frac{1}{2} t_2^4 - R = \left(t_1^3 + h\right) + \frac{1}{2}\left(t_2^3 - 2h\right) - R \\
&= \left(p^*_{(1,0,0,...,0)} + R\right) + \frac{1}{2}\left(p^*_{(1,1,0,...,0)} - p^*_{(1,0,0,...,0)}\right) - R = p^*_K.
\end{aligned}$$

This menu is $h$ first order strictly better than an optimum of simple menus. Moreover, this menu is a subset of $u^h$ and $u^h$ would extract the same expected profits from a population of buyer types distributed according to $\tilde{G}$.

Let us take $\varepsilon > 0$ and define the distribution $G_\varepsilon$ to be the convex combination of four normal distributions $N\left(t^i + \varepsilon h, (\varepsilon h)^4\right)$ restricted to $[0,1]^n$; the weights are $\mu\left(t^i\right)$, $i = 1, ..., 4$. As $\varepsilon \to 0+$ the expected seller's profit from $u^h$ over $G_\varepsilon$ approximates the expected profit from $u^h$ over $\tilde{G}$. Moreover, $\limsup_{\varepsilon \to 0+}$ of the maximum expected profit from a simple menu $u$ over $G_\varepsilon$ approximates the expected profit from $u$ over $\tilde{G}$, and the convergence is uniform over simple menus $u$ and over $h > 0$. Hence there is $\varepsilon > 0$ such that the expected profit from $u^h$ over $G_\varepsilon$ is $h$ first order better than the expected profit from an optimal simple menu over $G_\varepsilon$.

To end the proof, consider $(1-\alpha)\,F + \alpha G$ for small positive $\alpha$. Since $u^*$ and $u^h$ are $h$ first order equivalent on $F$ so $u^h$ weakly $h$ first order dominates any simple menu on $F$. On the other hand $u^h$ is $h$ first order strictly better than any simple menu on $G$. Thus for any $\alpha > 0$ the menu $u^h$ first order in $h$ strictly dominates any simple menu on $(1-\alpha)\,F + \alpha G$. This completes the proof of the density.

The proof of openness relies on Rochet's (1985) reformulation of the monopoly problem. In (1) the monopoly maximizes $\int p\,(t)\,dF\,(t)$ over pricing policies $(p, q)$. The maximization is constrained by the individual rationality and incentive compatibility of the buyers. Using incentive compatibility one can replace the individual rationality of the buyer by the assumption of zero price for a zero amount of both goods. Denote by $M_n$ the set of Lebesgue absolutely continuous probability distributions on $[0,1]^n$. As shown by Rochet for $F \in M_n$ the incentive constraints are equivalent to the convexity of the utility function $u\,(t) = tq\,(t) - p\,(t)$. Whenever $u$ is convex it is differentiable almost everywhere, and has one-sided partial derivatives everywhere on the interior of its domain. Denote by $\frac{\partial^+}{\partial t_1}$ the right-hand side derivative operator and by $\nabla$ the gradient operator[19]

$$\nabla u(t) = \left(\frac{\partial^+}{\partial t_1}u, \frac{\partial^+}{\partial t_2}u\right)(t)$$

Given the indifference-breaking assumption that indifferent buyers behave in a way preferred by the seller, the utility-maximizing quantity bought by a buyer is

$$q(t) = \nabla u(t).$$

and the price that the buyer pays is

$$p(t) = t\nabla u(t) - u(t).$$

Hence for $F \in M_n$, the monopoly problem translates into maximizing

$$\int t\nabla u(t) - u(t)\, dF \tag{8}$$

subject to $u(0,0) = 0$, $u$ is convex, and $\nabla u \in [0,1]^n$.

Denote by $U \subset C[0,1]^n$ the set of functions satisfying the constraints of (8). Since $U$ is a closed subset of the compact space $C[0,1]^n$ so $U$ is compact in the metrics inherited from $C[0,1]^n$. The proof of openness relies on the following result (proved in the appendix).

**Lemma 4.2.** The mapping

$$U \times M_n \ni (u,F) \to u \circ F = \int t\nabla u(t) - u(t)\, dF \in R$$

is continuous.

*Proof of the openness part of Theorem 4.1.* Use Rochet (1985) and consider the equivalent program (8). The compactness of $U$ and Lemma 5.2 allow us to invoke the Berge's Maximum Theorem (Berge (1963, p. 116)) to conclude that

$$M_n \ni F \to P(F) = \arg\max_{u\in U} \int t\nabla u(t) - u(t)\, dF$$

is upper hemicontinuous. Consequently, since $U^d$ is closed in $U$, so $P^{-1}(U^d)$ is closed in $M_n$, and thus the set of complex distributions is open in $M_n$.

[19] Alternatively we could work with the standard gradient that exists almost everywhere.

## 5. How Much Is Lost By the Restriction to Simple Bundling?

This section starts with an estimate of the worst-case scenario for a seller restricted to screening through simple product lines when there are two buyer types. It then constructs examples to show that the result of Section 4 — that in an optimal screening the seller may randomize over one attribute only — does not generalize to cases with three or more buyer types.

**Theorem 5.1.** If there are only two buyer types then the maximum percentage loss from the restriction to simple contracts is 12.5% of the best simple menu revenue. If $n$ is the number of attributes, then this bound is achieved for two types $A$,$B \in [0,1]^n$ such that

$$\mu_A = \mu_B = \frac{1}{2},$$
$$b_n + ... + b_2 = b_1 = \frac{1}{3},$$
$$a_1 = 1,\ a_2 = ... = a_n = 0.$$

It is easy to verify that for the parameters provided the loss is 12.5%. Let us prove that this is the maximal loss resulting from the restriction to simple contracts.

*Proof of Theorem 5.1.* Denote by $\pi^C$ the optimal profit and by $\pi^S$ the optimal profit from simple contracts. Our problem is to maximize $\frac{\pi^C - \pi^S}{\pi^S}$, or equivalently $\frac{\pi^C - \pi^S}{\pi^C}$, over two-type distributions $(a,b) = ((a_1,...,a_n),(b_1,...,b_n))$. By compactness and continuity of the problem the maximum exists. Notice that we can assume (2), and choose notation so that (3) and (4) are satisfied. Furthermore, by Theorem 3.1, we can restrict attention to situations when $b_n > a_n$ and $n^* < n^{**}$. Thus, the constraints on our problem are $a_i, b_i, \mu_A \in [0,1]$, (2), (3), (4), $b_n > a_n$, and $n^* < n^{**}$.

By Theorem 3.1, the optimal contract brings

$$\pi^C = \mu_A (a_1 + ... + a_n) + (1 - \mu_A)(\pi b_{n^*} + b_{n^*+1} + ... + b_n).$$

Let us estimate the profits $\pi^S$ from the optimal simple contract from below. The following two contracts are individually rational and incentive compatible:

- $\left\{ (1,...,1; a_1 + ... + a_n),\ \left( \underbrace{0,...,0}_{n^*}, \underbrace{1,...,1}_{n-n^*}; b_{n^*+1} + ... + b_n \right) \right\}$, or

- $\left\{ (1, ..., 1; a_1 + ... + a_n + (b_{n^*} - a_{n^*})), \left( \underbrace{0, ..., 0}_{n^*-1}, \underbrace{1, ..., 1}_{n-n^*+1}; b_{n^*} + ... + b_n \right) \right\}$.

The first of these two contracts brings

$$\pi^{s1} = \mu_A (a_1 + ... + a_n) + (1 - \mu_A)(b_{n^*+1} + ... + b_n)$$

and the second one brings

$$\pi^{s2} = \mu_A (a_1 + ... + a_n + (1 - \pi)[b_{n^*} - a_{n^*}]) + (1 - \mu_A)(b_{n^*} + ... + b_n).$$

Hence, $\frac{\pi^C - \max\{\pi^{S1}, \pi^{S2}\}}{\pi^C}$ is an upper bound on $\frac{\pi^C - \pi^S}{\pi^C}$.

Let us drop the constraint $n^* < n^{**}$ and maximize the upper bound $\frac{\pi^C - \max\{\pi^{S1}, \pi^{S2}\}}{\pi^C}$ subject to the remaining constraints. This is done via two claims proved in the appendix.

Claim 1. The maximum of the auxiliary problem with any $n$ is not higher than the maximum of the auxiliary problem with $n = 2$ and $n^* = 1$.

Claim 2. The maximum of the auxiliary problem with $n = 2$ and $n^* = 1$ is $\frac{1}{9}$.

It remains to verify that the upper bound $\frac{\pi^C - \max\{\pi^{S1}, \pi^{S2}\}}{\pi^C} = \frac{1}{9}$ is achievable for any $n \geq 2$ in the original problem. This upper bound is indeed achieved for the parameters stated in the theorem. This completes the proof.

In Section 3, we noted that when there are only two buyer types then it is enough for the seller to randomize over one attribute. This last corollary is false when there are more than two buyer types. A simple counterexample that violates this property has $n = 3$ and three buyer types

$$A = (1, 0, 0), B = (0, 1, 0), C = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

with masses $\mu_A, \mu_B, \mu_C$ such that $\mu_C << \mu_A, \mu_B$. Then, in any optimal menu of contracts the probability of allocating good 1 to type $C$ and the probability of allocating good 2 to type $C$ belong to $(0, 1)$.

The following example constructs a two-dimensional situation in which some buyers are allocated only lotteries.

**Example 5.2.** Consider $n = 2$ and three types $A = (a_1, a_2) = \left(1, \frac{1}{4}\right)$, $B = (b_1, b_2) = \left(0, \frac{3}{2}\right)$, $C = \left(\frac{1}{2}, \frac{1}{2}\right)$ that occur with probabilities $\mu_A, \mu_B, \mu_C$ such that $\mu_C << \mu_B << \mu_A$. Then, type $C$ buys a good with lotteries for both attributes.

Indeed, the optimal menu of contracts leads to types $A$ and $B$ being allocated the good $(1, 1)$ at the price $a_1 + a_2 = \frac{5}{4}$ because of the assumed condition on probabilities and $b_1 + b_2 > a_1 + a_2$. Conditional on this allocation, the incentive compatibility of $A$ and $B$ precludes the seller from selling any full attribute to type $C$. The seller can, however, offer the contract $\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{4}\right)$. $C$ will take this offer while $A$ and $B$'s incentive constraints will not be violated.

## 6. Conclusion

This paper contributes to our understanding of multidimensional screening. It solves the McAfee-McMillan (1988) problem when there are two buyer types and an arbitrary number of attributes. It then shows that, for a generic Lebesgue absolutely continuous distribution of buyer types, simple bundling strategies are suboptimal. Finally, the paper qualifies this last result in the case of two buyer types by computing the maximum difference between profits from the optimal seller's strategy and the optimal simple bundling strategy.

## References

Adams, William James, and Yellen, Janet L. (1976), "Commodity bundling and the burden of monopoly," *Quarterly Journal of Economics* 90, 475-498.

Ambjørnsen, Terje (2002), "Destructive Price Discrimination," mimeo, University of Oslo.

Armstrong, Mark (1996) "Multiproduct Nonlinear Pricing," *Econometrica* 64, 51-76.

Armstrong, Mark, and Jean-Charles Rochet (1999). "Multidimensional Screening: A User's Guide," *European Economic Revue* 43, 959-979.

Avery, Christopher, and Terrence Hendershott (2000), "Bundling and optimal auctions of multiple products," *Review of Economic Studies* 67, 483-497.

Berge, Claude (1963) *Topological Spaces*, translated by E. M. Patterson, New York: MacMillan.

Bulow, Jeremy and John Roberts (1989), “The Simple Economics of Optimal Auctions,” *Journal of Political Economy* 97, 1060-1090.

Court, Pascal and Li Hao (2000), “Sequential Screening,” *Review of Economic Studies* 67, 697-717.

Deneckere Raymond J., and Preston A. McAfee (1996), “Damaged Goods,” *Journal of Economics and Management Strategy* 5,149-174.

Figueroa, Nicolas, and Vasiliki Skreta (2006) “The Role of Outside Options in Auction Design,” manuscript.

Fudenberg, Drew and Jean Tirole (1983),“Sequential Bargaining with Incomplete Information,” *Revue of Economic Studies* 50, 221-247.

Kendall, Alexander (2004), “Stochastic Bundling with and without a Secondary Market,” honors thesis, Stanford University.

Manelli, Alejandro M., and David R. Vincent (2006) “Bundling as an Optimal Selling Mechanism for a Multiple-Good,” *Journal of Economic Theory* ....

Manelli, Alejandro M., and David R. Vincent (2004) “Optimal Pricing in a Multiple-Good Monopoly,” forthcoming in *Journal of Economic Theory*

McAdams, David (1998), “Multiproduct Monopoly Bundling,” manuscript.

McAfee, R. Preston, and John McMillan (1988), “Multidimensional Incentive Compatibility and Mechanism Design,” *Journal of Economic Theory* 46, 335-354.

McAfee, R. Preston, John McMillan, and Michael D. Whinston (1989), “Multiproduct Monopoly, Commodity Bundling, and Correlation of Values,” *Quarterly Journal of Economics* 104, 371-383.

Miller, Nolan, Nikita Piankov, and Richard Zeckhauser (2001), “When to Haggle,” Working Paper, Kennedy School of Government, Harvard University.

Mussa, Michael, and Sherwin Rosen (1978), “Monopoly and Product Quality,” *Journal of Economic Theory* 18, 301-317.

Myerson Roger B. (1981), “Optimal Auction Design,” *Mathematics of Operations Research* 6, 58-73.

Palfrey, Thomas R. (1983), “Bundling Decisions by a Multiproduct Monopolist with Incomplete Information,” *Econometrica* 51, 483-484.

Pycia, Marek (2000), “Genericity of Stochastic Mechanisms in Multidimensional Screening,” Part 2 of DEEQA Research Project, Toulouse University.

Rasul, Imran, and Silvia Sonderegger (2001), "Randomization with Countervailing Incentives," mimeo, London School of Economic.

Riley, John, and Richard Zeckhauser (1983) "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *The Quarterly Journal of Economics* 98 (2), 267-289.

Rochet, Jean-Charles (1985), "The taxation principle and multi-time Hamilton-Jacobi equations," *Journal of Mathematical Economics* 14, 113-128.

Rochet, Jean-Charles, and Philippe Choné (1998) "Ironing, Sweeping, and Multidimensional Screening," *Econometrica* 66, 783-826.

Rochet, Jean-Charles and Lars Stole (2003), "The Economics of Multidimensional Screening," *Advances in Economics and Econometrics: Theory and Applications - Eight World Congress*, M. Dewatripont, L. P. Hansen and S. J. Turnovsky (eds.), series Econometric Society Monographs, n. 36, Cambridge University Press.

Stokey, Nancy L. (1979), "Intertemporal Price Discrimination," *Quarterly Journal of Economics* 93, 355-371.

Thanassoulis, John (2004), "Haggling over Substitutes," *Journal of Economic Theory* 117, 217-245.

## Appendix

**Proof of Theorem 3.1.** A general form of the product line consists of two contracts $\tilde{A} = \left(q_1^A, ..., q_n^A; p^A\right)$ chosen by type $A$ and $\tilde{B} = \left(q_1^B, ..., q_n^B; p^B\right)$ chosen by type $B$. The optimal menu of contracts exists by the Weierstrass maximum theorem because prices $p^A, p^B \in [0, n]$ and hence the menu of contracts corresponds to a point in the compact set

$$([0,1]^n \times [0,n]) \times ([0,1]^n \times [0,n])$$

and the function from menus to the profits they generate is upper hemicontinuous. Similarly, there exists an optimal simple menu of contracts.

Let us first consider the problem of Theorem 4.1 and make two assumptions. First, notice that it is enough to consider strictly positive

$$a_i, b_i > 0, i = 1, ..., n.$$

Indeed, if we prove the claim in this case then the continuity of the expected profits from a fixed menu of contracts with respect to $a_1, ..., a_n, b_1, ..., b_n$ establishes the claim in the general case. Second, let us focus on the case

$$b_n > a_n$$

because the case of $b_n \leq a_n$ is straightforward.

There are two incentive constraints and two individual rationality constraints in the seller's maximization. Let us refer to them as IC-A, IC-B, IR-A, IR-B, where the labels are self-explanatory. Let us call a constraint slack if it may be dropped from the maximization and tight otherwise. Let us call a constraint strictly slack if it is satisfied with strict inequality in optimal menu of contracts, and weakly tight otherwise.

Before characterizing the optimal menu of contracts $\left(\tilde{A}, \tilde{B}\right)$ let us prove four claims.

Claim 1. If IC-B is weakly tight, then $q_i^B = q_i^A = 1$ for $i = 1, ..., n$, and $p^A = p^B = b_1 + ... + b_n$.

To prove Claim 1 first note that if IC-B is weakly tight then also IC-A is weakly tight. Indeed, if $A$ strictly preferred $\tilde{A}$ to $\tilde{B}$, then we would have $q_i^B = 1$ for $i = 1, ..., n$ because $q_i^B < 1$ for some $i = 1, ..., n$ would allow the seller to gainfully replace the contract $\tilde{B}$ with

$$\left(q_1^B, ..., q_i^B + \varepsilon, ..., q_n^B; p^B + b_i\varepsilon\right)$$

for some small positive $\varepsilon$. Consequently, $A$'s strict preference of $\tilde{A}$ over $\tilde{B}$ would imply that $p^A < p^B$. But this is a contradiction, as then the Seller would be better off by proposing single contract $\tilde{B} = \left(1, ..., 1; p^B\right)$ that would be accepted by both $A$ and $B$. Thus, if IC-B is weakly tight then both $A$ and $B$ are indifferent between the contracts $\tilde{A}$ and $\tilde{B}$. Hence, also the seller is indifferent between selling either of the products, and this means that $p^A = p^B \leq b_1 + ... + b_n$. The optimal among such contracts sets $q_i^B = q_i^A = 1$ for all $i = 1, ..., n$ and $p^A = p^B = b_1 + ... + b_n$, which proves the claim.

Claim 2. If IC-B is strictly slack then
(a) $q_i^A = 1$ for $i = 1, ..., n$,
(b) IR-B is weakly tight, and $p^B = b_1 q_1^B + ... + b_n q_n^B$,
(c) IC-A is weakly tight and

$$\begin{aligned} p^A &= p^B + \left(1 - q_1^B\right) a_1 + ... + \left(1 - q_n^B\right) a_n \\ &= a_1 + ... + a_n - (a_1 - b_1) q_1^B - ... - (a_n - b_n) q_n^B \end{aligned}$$

To prove (a) notice that if $q_i^A < 1$ for an $i = 1, ..., n$, then the Seller could do better by replacing $\tilde{A}$ by $\left(q_1^A, ..., q_i^A + \varepsilon, ..., q_n^A; p^A + a_i \varepsilon\right)$ for some small positive $\varepsilon$. To show (b) notice that with IR-B strictly slack the seller could benefit by raising $p^B$. To show (c) notice that with IC-A strictly slack, the seller would profitably increase any $q_i^B$ that is smaller than 1. However, $q_i^B = 1$ for all $i = 1, ..., n^A$ cannot obtain as it would mean that the bundles offered to both types are identical, and by the incentive compatibility conditions their prices would have to be equal, and thus type $B$ buyer would be indifferent between the contracts, contrary to the strict slackness of IC-B. Since IC-A is weakly tight, the formula for $p^A$ follows from (a).

Denote

$$d_i = a_i - b_i.$$

By Claim 2, if IC-B is strictly slack then the seller maximizes

$$\begin{aligned} &\max_{q_i^B \in [0,1]} \mu_A p^A + \mu_B p^B \\ &= \mu_A \left(a_1 + ... + a_n - d_1 q_1^B - ... - d_n q_n^B\right) + \mu_B \left(b_1 q_1^B + ... + b_n q_n^B\right) \\ &= \left(\mu_B b_1 - \mu_A d_1\right) q_1^B + ... + \left(\mu_B b_n - \mu_A d_n\right) q_n^B + \text{constant} \end{aligned}$$

subject to A's participation constraint, IR-A,

$$a_1 + ... + a_n - d_1 q_1^B - ... - d_n q_n^B = p^A \leq a_1 + ... + a_n.$$

Consequently, when IC-B is slack then the seller's problem can be stated as

$$\max_{q_i^B \in [0,1]} (b_1 - \mu_A a_1) q_1^B + ... + (b_n - \mu_A a_n) q_n^B \tag{(P)}$$

subject to IR-A constraint

$$d_1 q_1^B + ... + d_n q_n^B \geq 0.$$

Claim 3. Assume that IC-B is strictly slack. The following conditions are equivalent:
(a) IR-A is slack in the maximization (P),
(b) $d_{n^{**}} + ... + d_n \geq 0$,

Indeed, unconstrained by IR-A the seller would set $q_i^B = 1$ whenever $\frac{b_i}{a_i} > \mu_A$, $q_i^B = 0$ whenever $\frac{b_i}{a_i} < \mu_A$, and be indifferent what values are taken by $q_i^B$ whenever $\frac{b_i}{a_i} = \mu_A$. Since $n^{**}$ denotes the minimum $i$ such that $\frac{b_i}{a_i} \geq \mu_A$, so in an IR-A unconstrained optimal contract $q_1^B = ... = q_{n^{**}-1}^B = 0$ and $q_{n^{**}}^B = ... = q_n^B = 1$. Thus, the slackness of IR-A is equivalent to (b).

Claim 4. The following conditions are equivalent:
(a) $d_{n^{**}} + ... + d_n \geq 0$,
(b) $n^{**} \leq n^*$,
(c) $\mu_A \leq \frac{b_{n^*}}{a_{n^*}}$.

Recall that $b_n > a_n$, and thus $n^{**} \in \{1, ..., n\}$. To see the equivalence of (a) and (b) note that the monotonicity of $\frac{b_i}{a_i}$ and definition of $n^*$ implies that $d_i + ... + d_n \geq 0$ iff $i \in \{1, ..., n^*\}$. To see the equivalence of (b) and (c) note that the monotonicity of $\frac{b_i}{a_i}$ and definition of $n^{**}$ implies that $\frac{b_i}{a_i} \geq \mu_A$ iff $i \in \{n^{**}, ..., n\} \cup \{+\infty\}$.

Now, we are ready to solve the seller's problem separately considering $n^{**} \leq n^*$ and $n^* < n^{**}$.

Case $n^{**} \leq n^*$. Either IC-B is strictly slack or weakly tight. This gives two potential solutions. By Claims 3 and 4, the solution for IC-B strictly slack may be obtained by solving unconstrained (P) and is written out in Theorem 4.1. The solution for IC-B weakly tight is given in Claim 1. It remains to check that the solution in Theorem 4.1 is weakly better than the solution in Claim 1.

The difference in expected profits between the solutions is

$$\begin{aligned}&\mu_A\left(a_1+...+a_{n^{**}-1}+b_{n^{**}}+...+b_n\right)+\mu_B\left(b_{n^{**}}+...+b_n\right)-\left(b_1+...+b_n\right)\\&=\mu_A\left(a_1+...+a_{n^{**}-1}\right)-\left(b_1+...+b_{n^{**}-1}\right).\end{aligned}$$

Since $a_n < b_n$, we have $n^{**} < +\infty$. If $n^{**} \in \{2, ..., n\}$, then this difference is strictly positive because $\mu_A > \frac{b_{n^{**}-1}}{a_{n^{**}-1}} \geq \frac{b_i}{a_i}$ for $i \leq n^{**}-1$ by definition of $n^{**}$ and monotonicity of $\frac{b_i}{a_i}$. In particular then IC-B is indeed strictly slack. If $n^{**} = 1$, then the two solutions are identical (and IC-B is weakly tight).

Case $n^* < n^{**}$. Then IR-A is tight, and thus (P) reduces to

$$\max_{q_i^B\in[0,1]}\left(b_1-\mu_A a_1\right)q_1^B+...+\left(b_n-\mu_A a_n\right)q_n^B=\left(1-\mu_A\right)\left(b_1q_1^B+...+b_nq_n^B\right)$$

subject to

$$d_1q_1^B+...+d_nq_n^B=0.$$

Thus, there exists $k \in [0,1]$ such that

$$\begin{aligned}q_i^B&=1\text{ whenever }\frac{b_i}{d_i}>\frac{1-k}{k},\\q_i^B&=0\text{ whenever }\frac{b_i}{d_i}<\frac{1-k}{k},\end{aligned}$$

and $q_i^B$ for $i$ such that $\frac{b_i}{d_i} = \frac{1-k}{k}$ are determined by the constraint, not necessarily in a unique way. Equivalently

$$\begin{aligned}q_i^B&=1\text{ whenever }\frac{b_i}{a_i}>k\\q_i^B&=0\text{ whenever }\frac{b_i}{a_i}<k\end{aligned}$$

and $q_i^B$ for $i$ such that $\frac{b_i}{a_i} = k$ are determined by the constraint. Note that $a_i > 0$ for $i = 1, ..., n^A$ and that $k \leq \mu_A$. There is some indeterminacy for $i$ such that $\frac{b_i}{a_i} = k$. Without loss of generality we can assume that $q_i^B = 0$ or 1 for all such $i$ except for one, let us call it $n^{***}$ and choose it in such a way that

$$\begin{aligned}q_i^B&=1\text{ for }i>n^{***}\\q_i^B&=0\text{ for }i<n^{***}\end{aligned}$$

and

$$q^B_{n^{***}} = \frac{-\left(d_{n^A+1} + ... + d_n\right) - \left(d_{n^{***}+1} + ... + d_{n^A}\right)}{d_{n^{***}}} = \frac{-d_{n^{***}+1} - ... - d_n}{d_{n^{***}}}.$$

and $q^B_{n^{***}} \in (0,1]$.By definition of $n^*$, this properties imply that $n^{***} = n^*$. Hence,

$$\pi = q^B_{n^*} = \frac{-d_{n^*+1} - ... - d_n}{d_{n^*}} \in (0,1],$$

and the solution is as postulated in Theorem 4.1. It remains to check that this solution is preferred by the seller to the optimal solution with IC-B tight (described in Claim 1); the slackness of IC-B will be then automatically satisfied. The difference in expected profits from the two solutions is

$$\begin{aligned}
&\mu_A\left(a_1 + ... + a_n - d_1 q^B_1 - ... - d_n q^B_n\right) + \mu_B\left(b_1 q^B_1 + ... + b_n q^B_n\right) - \left(b_1 + ... + b_n\right) \\
&= \mu_A\left(a_1 + ... + a_n\right) + \mu_B\left(b_{n^*}\pi + b_{n^*+1} + ... + b_n\right) - \left(b_1 + ... + b_n\right) \\
&= \mu_A\left(a_1 + ... + a_{n^*} - d_{n^*}\pi\right) + \mu_B b_{n^*}\pi - \left(b_1 + ... + b_{n^*}\right) \\
&= \mu_A\left(a_1 + ... + a_{n^*} - \left(a_{n^*} - b_{n^*}\right)\pi\right) + \left(1 - \mu_A\right) b_{n^*}\pi - \left(b_1 + ... + b_{n^*}\right) \\
&= \mu_A\left(a_1 + ... + a_{n^*-1} + \left(1-\pi\right) a_{n^*}\right) - \left(b_1 + ... + b_{n^*-1} + \left(1-\pi\right) b_{n^*}\right)
\end{aligned}$$

and is strictly positive (as required) because $\mu_A > \frac{b_{n^*}}{a_{n^*}} \geq \frac{b_i}{a_i}$ for $i \leq n^*$. The genericity claim of Theorem 4.1 is straightforward. This completes the proof.

**Proof of Lemma 4.2**. First note for any $u \in U$ the function $\Phi(u)$ such that $\Phi(u)(t) = t\nabla u(t) - u(t)$ is well-defined as $\nabla u(t)$ exists everywhere. Note that $\Phi(u)$ is measurable, and it is bounded since $\nabla u(t) \in [0,1]^n$. Moreover, $\nabla u(t) \in [0,1]^n$ and $u(0) = 0$ for $t \in [0,1]^n$ imply that

$$\Phi(u)(t) = t\nabla u(t) - u(t) \in [-n, n]$$

for $t \in [0,1]^n$ and $u \in U$.

Take $(u, F) \in U \times M_n$ and a sequence $(u_k, F_k) \in U \times M_n$ that tends to $(u, F)$. We are to prove that $u_k \circ F_k \to u \circ F$. Note that

$$|u_k \circ F_k - u \circ F| \leq |u_k \circ F_k - u \circ F_k| + |u \circ F_k - u \circ F|$$

and thus it is enough to show the convergence to 0 of both elements of the left-hand-side sum.

Consider the first element of the sum, take a small $\varepsilon > 0$, and note that

$$\begin{aligned}
&|u_k \circ F_k - u \circ F_k| \\
&\leq \int_{[0,1]^n} |\Phi(u_k)(t) - \Phi(u)(t)|\, dF_k \\
&= \int_{[0,1-\varepsilon]^n} |\Phi(u_k)(t) - \Phi(u)(t)|\, dF_k + \int_{[0,1]^n - [0,1-\varepsilon]^n} |\Phi(u_k)(t) - \Phi(u)(t)|\, dF_k.
\end{aligned}$$

For any small $\varepsilon > 0$ the first integral tends to 0 as $k \to \infty$ because $u_k \to u$ uniformly and all those functions are convex. Moreover, by $\Phi(u)(t), \Phi(u_k)(t) \in [-n, n]$ the second integral is smaller than

$$2nF_k([0,1]^n - [0,1-\varepsilon]^n).$$

Since the weak convergence $F_k \to F$ implies the convergence

$$F_k([0,1]^n - [0,1-\varepsilon]^n) \to F([0,1]^n - [0,1-\varepsilon]^n),$$

$F$ is Lebesgue absolutely continuos, and the Lebesgue measure of $[0,1]^n - [0,1-\varepsilon]^n$ tends to 0 as $\varepsilon \to 0$, so the second integral can be shown to tend to 0 as $k \to \infty$ and $\varepsilon \to 0$. Taking this together we may conclude that

$$|u_k \circ F_k - u \circ F_k| \to 0$$

as $k \to \infty$.

It remains to show that $|u \circ F_k - u \circ F| \to 0$ with $k \to \infty$. This is so if $\Phi(u)$ is continuous. In general, since $u$ is continuous, it is enough to show that

$$\left| \int_{[0,1]^n} t\nabla u(t)\, dF_k - \int_{[0,1]^n} t\nabla u(t)\, dF \right| \to 0 \text{ as } k \to \infty$$

and furthermore, we can analyze the elements of the sum $t\nabla u(t)$ separately, so it is enough to show that

$$\left| \int_{[0,1]^n} t_i \frac{\partial^+}{\partial t_i} u(t)\, dF_k - \int_{[0,1]^n} t_i \frac{\partial^+}{\partial t_i} u(t)\, dF \right| \to 0 \text{ as } k \to \infty$$

for $i = 1, ..., n$. Denoting by $f$ and $f_k$ the densities of $F$ and $F_k$, respectively, we can write a sufficient condition for the above property as

$$\left| \int_{[0,1]} t_i \frac{\partial^+}{\partial t_i} u(t)\, f_k(t)\, dt_i - \int_{[0,1]} t_i \frac{\partial^+}{\partial t_i} u(t)\, f(t)\, dt_i \right| \to 0 \text{ as } k \to \infty$$

for $i=1,...,n$ and any $t_1,...,t_{i-1},t_{i+1},...,t_n\in[0,1]^{n-1}$. Now, $t_i\to t_i\frac{\partial^+}{\partial t_i}u(t)$ is increasing, so the set of discontinuities is of measure 0. Let $I$ be a union of intervals of total length $\varepsilon$ that covers the set of discontinuities. We can find a continuous $\varphi$ such that $\varphi(t_i)\in[0,1]$ for all $t_i$ and $\varphi(t_i)=t_i\frac{\partial^+}{\partial t_i}u(t)$ if $t_i\notin I$, and decompose

$$\begin{aligned}
&\left|\int_{[0,1]} t_i\frac{\partial^+}{\partial t_i}u(t)\,f_k(t)\,dt_i-\int_{[0,1]} t_i\frac{\partial^+}{\partial t_i}u(t)\,f(t)\,dt_i\right| \\
&\leq\left|\int_{[0,1]}\varphi(t_i)\,f_k(t)\,dt_i-\int_{[0,1]}\varphi(t_i)\,f(t)\,dt_i\right| \\
&+\left|\int_I\left(t_i\frac{\partial^+}{\partial t_i}u(t)-\varphi(t_i)\right)f_k(t)\,dt_i-\int_I\left(t_i\frac{\partial^+}{\partial t_i}u(t)-\varphi(t_i)\right)f(t)\,dt_i\right|.
\end{aligned}$$

Now, the first difference tends to 0 as $F_k\to F$ and $\varphi$ is continuos. Moreover, $F_k\to F$ implies also that there exists an $M$ independent of $I$ such that for large $k$, the second difference is smaller than $M\varepsilon$. This concludes the proof.

**Proof of Claims 1 and 2 from the proof of Theorem 5.1.**

Claim 1. The maximum of the auxiliary problem with any $n$ is not higher than the maximum of the auxiliary problem with $n=2$ and $n^*=1$.

Let us prove this claim in two steps. First notice that if $n^*+1<n$ then the lower dimensional problem with $n'=n^*+1$ attributes and valuations

$$\begin{aligned}
a_1'&=a_1,...,a_{n^*}'=a_{n^*},a_{n^*+1}'=a_{n^*+1}+...+a_n,\\
b_1'&=b_1,...,b_{n^*}'=b_{n^*},b_{n^*+1}'=b_{n^*+1}+...+b_n,
\end{aligned}$$

satisfies all constraints and attains the same objective $\frac{\pi^C-\max\left\{\pi^{S1},\pi^{S2}\right\}}{\pi^C}$.

Second, notice that if $n^*>1$ then the lower dimensional problem with $n'=n-n^*+1$ and

$$a_i'=a_{n^*-1+i},\ \ b_i'=b_{n^*-1+i},\ \text{for } i=1,...,n'$$

satisfies the constraints. Indeed, (3) is not violated because the definition of $n^*$ implies that

$$a_{n^*}+...a_n\geq b_{n^*}+...+b_n.$$

Other constraints are satisfied in a straightforward manner. This lower dimensional problem attains weakly higher objective $\frac{\pi^C-\max\left\{\pi^{S1},\pi^{S2}\right\}}{\pi^C}$ as the nominator does not change and the denominator weakly increases. This proves Claim 1.

Claim 2. The maximum of the auxiliary problem with $n = 2$ and $n^* = 1$ is $\frac{1}{9}$.

Without changing the maximum, we can add variable $\pi$ to the set of variables we maximize over, and add its definition (6) to the set of constraints. Thus, the problem takes the form

$$\max_{a_i, b_i, \mu_A, \pi} \frac{\pi^C - \max\left\{\pi^{S1}, \pi^{S2}\right\}}{\pi^C} = 1 - \frac{\max\left\{\pi^{S1}, \pi^{S2}\right\}}{\pi^C}$$
$$= 1 - \frac{\max\left\{\mu_A (a_1 + a_2) + (1 - \mu_A) b_2, \mu_A (a_1 + a_2 + (1 - \pi)[b_1 - a_1]) + (1 - \mu_A)(b_1 + b_2)\right\}}{\mu_A (a_1 + a_2) + (1 - \mu_A)(\pi b_1 + b_2)}$$
$$= \frac{(1 - \mu_A) \pi b_1 - \max\left\{0, \mu_A (1 - \pi)[b_1 - a_1] + (1 - \mu_A) b_1\right\}}{\mu_A (a_1 + a_2) + (1 - \mu_A)(\pi b_1 + b_2)}$$

subject to $a_i, b_i, \mu_A \in [0, 1]$, (2), (3), (4), (6), $b_2 > a_2$.

First, notice that at the maximum

$$0 = \mu_A (1 - \pi)[b_1 - a_1] + (1 - \mu_A) b_1. \tag{9}$$

Indeed, if $\mu_A (1 - \pi)[b_1 - a_1] + (1 - \mu_A) b_1 < 0$ at the maximum, then the objective could be made arbitrarily close to 1 by taking $\mu_A = 0$, $\pi \sim 1$ (i.e., $a_1 - b_1 \sim b_2 - a_2$), and $b_2 << b_1$. If $\mu_A (1 - \pi)[b_1 - a_1] + (1 - \mu_A) b_1 > 0$ at the maximum, then the objective could be made arbitrarily close to 1 by taking $a_2, b_1, b_2 \sim 0$ and $a_1 >> 0$ (note that then also $\pi \sim 0$).

Taking (9) into account, we can reduce the auxiliary problem to

$$\max_{a_1, a_2, b_1, b_2, \mu_A, \pi} \frac{(1 - \mu_A) \pi b_1}{\mu_A (a_1 + a_2) + (1 - \mu_A)(\pi b_1 + b_2)}$$

subject to $b_1 \in [0, 1]$, $a_1, b_2, \pi \in (0, 1]$, (6), (9), and $a_1 > b_1$.

Second, notice that the objective increases and all constraints are satisfied when we decrease $a_2$ and $b_2$ while maintaining (6). Thus, at the maximum, $a_2 = 0$, and (6) implies that

$$b_2 = \pi (a_1 - b_1).$$

At the same time, (9) implies that

$$\mu_A = \frac{b_1}{b_1 + (1 - \pi)[a_1 - b_1]}.$$

Plugging these two expressions into the maximization, we can reduce it further to

$$\max_{a_1, b_1, \pi} \frac{(1 - \pi)[a_1 - b_1] \pi b_1}{b_1 a_1 + (1 - \pi)[a_1 - b_1](\pi b_1 + \pi (a_1 - b_1))}$$
$$= \frac{(1 - \pi) \pi [a_1 - b_1] b_1}{b_1 a_1 + (1 - \pi) \pi [a_1 - b_1] a_1}$$

subject to $b_1 \in [0,1]\,, a_1, \pi \in (0,1]$, and $a_1 > b_1$.

Third, notice that $b_1 > 0$ at the maximum and thus we can simplify the problem further to

$$\max_{a_1,b_1,\pi} \frac{1}{\frac{1}{(1-\pi)\pi}\frac{a_1}{a_1-b_1} + \frac{a_1}{b_1}}$$

subject to $b_1, a_1, \pi \in (0,1]$, and $a_1 > b_1$. At the maximum, $\pi = \frac{1}{2}$. Substituting $x = \frac{a_1}{b_1}$ we reduce the problem to minimizing the denominator $f(x) = 4\frac{x}{x-1} + x$ over $x > 1$. The problem is convex as $f''(x) = \frac{8}{(x-1)^3} > 0$. Thus, the minimum is achieved at $f'(x) = 4\frac{-1}{(x-1)^2} + 1 = 0$, that is at $x = 3$. The minimum equals $\frac{1}{4\frac{3}{2}+3} = \frac{1}{9}$ and Claim 2 is proved.