Many-to-One Matching with Complementarities and Peer Effects

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Abstract

This paper studies many-to-one matching problems such as between students and colleges, and workers and firms in the general case, in which both peer effects and complementarities are allowed. In a matching, an agent on one side, say a firm, employs a subset of agents from the other side (workers), thus forming a coalition. The paper interprets an agent's payoff in a matching as determined by a division rule applied to the value created by the agent's coalition. The main results relate stability to pairwise alignment. A matching is stable if no group of agents can profitably deviate. Agents' preferences are pairwise aligned if any two agents in the intersection of any two coalitions prefer the same one of the two coalitions. The results say that under mild regularity conditions (i) if the division rule generates pairwise-aligned preferences then there exists a stable matching, and (ii) if there exists a stable matching for all profiles of coalitional values then the division rule generates pairwise-aligned preferences. The Nash bargaining and Tullock rent-seeking are examples of division rules that satisfy the proposed pairwise-alignment condition and were not previously linked to stability.

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1. Introduction

This paper studies many-to-one matching problems such as between students and colleges, interns and hospitals, and workers and firms.[1] An agent on one side, say a firm, can hire as many workers as it needs, and an agent on the other side, a worker, can be employed by a firm or remain unemployed. In this way, the agents form coalitions: (i) an unemployed worker is considered a coalition, and (ii) all other coalitions consist of a firm and its workforce. Each agent has preferences over the coalitions that contain this agent. A matching is stable if (i) no matched worker prefers unemployment to working for the matched firm, and (ii) no firm can hire a subset of workers to whom that firm is matched and a group (possibly empty) of additional workers so that the firm and all the additional workers strictly increase their payoffs. Starting with the work of Roth (1984) on US matching between interns and hospitals, substantial empirical evidence links the lack of stability with market failures.[2]

The most general known sufficient conditions for stability are derived from the Kelso and Crawford (1982) gross-substitutes condition.[3] In a formulation of Roth and Sotomayor (1990), the substitutability condition is as follows: if a firm wants to employ a worker w from a large pool of workers, then the firm wants to employ w from any smaller pool containing w. Kelso and Crawford (1982) show that if firms' preferences satisfy the substitutability condition and there are no peer effects – that is, workers' preferences depend only on the firm they apply to and not on who their peers will be – then there exists a stable many-to-one matching.

There are matching settings that do not satisfy the standard assumptions of substitutability and lack of peer effects. The substitutability condition fails if there are

[1] The college admission problem was introduced by Gale and Shapley (1962). A recent example from the realm of education is the design of a new high school admissions system in New York City, which allows both schools and students to influence the matching (Abdulkadiroğlu, Pathak, and Roth 2005). Medical labor markets are studied for example in Roth (1984), Roth (1991), Roth and Peranson (1999), Niederle and Roth (2004), and McKinney, Niederle, and Roth (forthcoming). Roth (2002) provides a survey. Roth and Sotomayor (1990) is a classic survey of theory, empirical evidence, and design applications of the many-to-one matching models.

[2] Gale and Shapley (1962) raised the question of stability of matchings. The evidence linking lack of stability to market failures is surveyed in Roth and Sotomayor (1990) and Roth (2002).

[3] Cf. Roth and Sotomayor (1990), Echenique and Oviedo (2006), Hatfield and Milgrom (2005), and Ostrovsky (2005). Roth (1985)'s responsiveness condition is also a variant of substitutability.

non-trivial complementarities between workers. It also fails when there are fixed costs. The complementarities are non-trivial if, for example, a firm's production process is profitable only when adequately staffed. For instance, a biotech firm may not open a new R&D lab if it is unable to hire experts in all complementary areas required for the lab's work. Substitutability fails for firms with fixed costs if their operations must be of some minimal size to ensure profitability. Peer effects are present if workers care about interactions in the workplace or if the identity of other workers non-trivially influences workloads or day-to-day bargaining between workers.

This paper provides a novel sufficient and, in a sense explained below, necessary condition for stability that may be used to analyze settings with complementarities and peer effects such as those mentioned above. The paper also shows that the condition is satisfied by several standard models of economic interactions that have not previously been recognized as admitting stable matchings.

The main component of the proposed condition is the pairwise alignment of preferences. Agents' preferences are *pairwise aligned* if the restrictions of any two agents' preference relations to the set of coalitions to which they both belong coincide. For instance, a firm and a worker either both prefer to form a firm-and-one-employee coalition or both prefer a larger coalition that includes the firm, the worker, and some other workers.

The sufficient and necessary condition is developed in three stages, from specific to more general environments.

Section 2 presents an example of matching with payoffs to members of any possible coalition being determined by equal division of the revenue (or, more abstractly, of the transferable-utility value) that the coalition generates. Section 4 generalizes this example by replacing the equal division rule with a broad class of division rules. The results are particularly relevant for matching situations in which agents are unable to negotiate binding agreements. Section 5 addresses the problem in a nontransferrable-utility setting with agents' preferences as primitives.

In the example of Section 2, there are two dates. On date 1, firms and workers match – that is, form coalitions. On this date, firms and workers cannot negotiate binding employment contracts. In effect, on date 1, the agents' preferences over coalitions result from the agents' expectations of the payoffs that will be negotiated on date 2. On date 2, each coalition creates a value and its members divide the value equally among themselves.

Each resultant preference profile is pairwise aligned and, thus, the pairwise alignment condition is embedded in this setting.

Section 2 shows that there is a stable matching in the equal division example. It also proves a stronger property of this matching setting, namely the existence of a metaranking. A metaranking is a transitive relation on all coalitions; its defining property is that, restricted to coalitions containing an agent, the transitive relation agrees with preferences of this agent.[4]

Section 4 discusses matching when payoffs are determined by division rules that satisfy mild regularity properties. A division rule may represent a continuation game or a bargaining protocol. As in the example of Section 2, each coalition has a value. The division rule takes the values of coalitions, that is the value function, and generates agents' payoffs and resultant preferences over coalitions. A division rule is called pairwise aligned if agents' preferences over coalitions are pairwise aligned for any value function.

Section 4 establishes a sufficient and necessary condition for stability. If agents' payoffs are determined by a pairwise-aligned division rule then there exists a stable matching. Furthermore, if a division rule generates payoffs such that there exists a stable matching for any value function, then the division rule is pairwise aligned.

Section 5 reformulates the problem with agents' ordinal preferences over coalitions as primitives. The sufficient condition imposes pairwise alignment on agents' preferences from a rich domain of preference profiles as it is not sufficient for stability to impose pairwise alignment on a single preference profile (an example of a matching situation with pairwise-aligned preferences and no stable matching is included in Section 4). In the preference framework of Section 5, the pairwise alignment remains a necessary condition for the existence of stable matchings for all preference profiles from large domains of profiles.

Section 5 also relaxes the metaranking property in such a way that (i) it is satisfied in the Gale and Shapley (1962) marriage model, (ii) it is satisfied whenever payoffs are generated by a pairwise-aligned division rule, and (iii) it is equivalent to the existence of a rich domain of pairwise-aligned preference profiles. Section 5 then shows that if agents' preferences satisfy the relaxed metaranking property then there exists a stable matching.

The sufficiency and necessity results proved in this paper allow one to determine which

[4]This example mimics Farrell and Scotchmer's (1988) study of partnerships that share the surplus equally among their members. They showed that the core is non-empty in a coalition formation game followed by an equal division of value. They also introduced the concept of a metaranking.

sharing rules and games induce the existence of stable matchings. Section 4 shows that the Nash bargaining solution always induces stability while the Kalai and Smorodinsky bargaining solution does not. Section 6 determines the class of linear sharing rules and the class of welfare-maximization mechanisms that induce the existence of stable matchings. Section 6 also shows that there is always a stable matching if agents' preferences are induced by Tullock's (1980) rent-seeking game.

The two main ideas of the present paper – the framework of division rules for the analysis of stability and the pairwise-alignment condition for stability – are both new. As discussed above, the many-to-one matching literature – e.g., Gale and Shapley (1962), Kelso and Crawford (1982), Roth (1985), Roth and Sotomayor (1990), Hatfield and Milgrom (2005) – has used variants of the assumptions of substitutability and of the lack of peer effects. To the best of my knowledge there are only four papers that move beyond one or both of these assumptions: Dutta and Massó (1997), Revilla (2004), Echenique and Oviedo (2006), and Echenique and Yenmez (2007).

Dutta and Massó (1997) maintained the substitutability condition and weakened the lack of peer effects condition in two ways: (i) allowing exogenously "married" workers to prefer any coalition that includes their partner to any other coalition, and (ii) allowing peer effects to influence workers' preferences between two coalitions if the employer (firm) is the same but not otherwise. Revilla (2004) generalized their first result by replacing worker couples with more general groups of workers. He also analyzed a situation in which agents' preferences are determined by an exogenous ranking of workers. The present paper goes beyond these two papers by (i) proposing a new, division-rule based, way to look at matching problems, (ii) proving a general existence condition that allow both complementarities and peer effects, (iii) proving the necessity results, and (iv) finding situations in which the proposed condition is satisfied and that were not previously recognized as admitting stable matchings.

Echenique and Oviedo (2006) and Echenique and Yenmez (2007) construct algorithms that find stable matchings in general settings. In contrast, the goal of this paper is to develop our understanding when stable matching exist.

In addition to the above-mentioned many-to-one matching papers, there are two papers on the existence of core coalition structure in (one-sided) coalitional games that are related to the present results. Farrell and Scotchmer (1988) study the formation of partnerships. They show that the core is non-empty in a coalition formation game followed by an equal division of value. An illustrative example presented in Section 2 is based on their insight. Farrell and Scotchmer (1988) introduce also the concept of metaranking that is being used in Sections 2 and 4.[5] Banerjee, Konishi, and Sönmez (2001) show that the equal division may be replaced by some other linear sharing rules in Farrell and Scotchmer's analysis.[6] Neither of these papers studies the stability properties of general division rules nor recognizes the connection between stability and the pairwise alignment.

All examples discussed in the present paper exhibit a two-date structure: on the first date agents match but are unable to negotiate binding contracts, on the second day, after the coalitions are locked in, the division of value is negotiated. This two-date structure has been used in an empirical paper by Sørensen (2005) who studies many-to-one matching between start-ups and (lead) venture capital firms, and quantifies the synergies between a start-up and the venture capital firm that is matched with it. He argues that due to severe contractual incompleteness start-ups and venture capital are unable to structure their future relationships in a binding way when they match. Hence, the profits are divided only as they are realized, after the matching is concluded. In contrast to the present paper, Sørensen assumes that there are no synergies between start-ups in a venture capital firm's portfolio. In particular, in Sørensen (2005) there are no complementarities (that is synergies between start-ups reflected in the venture-capital payoff) and there are no peer effects (that is synergies between start-ups reflected in start-ups' payoffs).

2. Example

Let us consider the following many-to-one matching problem. On date 1, firms and workers match, that is, form coalitions. On this date, firms and workers cannot enter binding employment contracts. In effect, on date 1, the agents' preferences over coalitions

[5] Metarankings are related to pairwise alignment. If a metaranking exists, then preferences are pairwise aligned. The converse is true in the special case studied in Section 4 but not in the general setting of Section 5. The main practical difference between the two concepts is that pairwise alignment may be directly verified, while to establish the existence of metaranking one needs to construct it.

[6] Their main result is a relaxation of the Farrell and Scotchmer metaranking property in a direction unrelated to the present paper. Their "top coalition property" says that each subgroup of agents contains a coalition that is weakly preferred by all its members to any other coalition of agents in the subgroup.

reflect the agents' expectations of the payoffs that will be determined on date 2. On date 2, each resultant coalition, C, creates value $\mathbf{v}(C) \geq 0$, and its members divide $\mathbf{v}(C)$ equally that is each agent $i \in C$ receives $\frac{\mathbf{v}(C)}{\#C}$ where $\#C$ is the number of agents in C.[7] This example includes situations with complementarities between workers as no assumption is made about the function $\mathbf{v}$. The peer effects are inherent to the equal division rule.

In the above matching problem there exists a stable matching. Recall that a matching is stable if (i) no matched worker prefers unemployment to working for the matched firm, and (ii) no firm can hire a subset of workers to whom that firm is matched and a group (possibly empty) of additional workers so that the firm and all the additional workers strictly increase their payoffs.[8]

To construct a stable matching, let us first observe that no agents would ever want to change a coalition that maximizes the index $\frac{v(C)}{\#C}$. Therefore, the coalition with maximal $\frac{v(C)}{\#C}$ may be treated as if its members did not participate in the matching between the remaining agents. In this way, one can recursively construct a stable matching.

This example leads to a question what property of the equal division rule guarantees the existence of stable matching, and, more generally, what division rules induce stability in matching problems. This question will be answered in Sections 4 and 5.

3. Basic Concepts

A finite set of agents I is divided into two non-empty disjoint sets, $I = F \cup W$. We will refer to agents from F as firms, and to agents from W as workers. Each worker seeks a firm, and each firm $f \in F$ seeks up to M_f workers, where $M_f \geq 1$. A matching is a function μ from $F \cup W$ into subsets of $F \cup W$, such that

- $\mu(w) = \{f\}$ if the worker w is employed by the firm f, and $\mu(w) = \{w\}$ if w is unemployed,
- $\mu(f) \subset W$ and the size $\#\mu(f) \leq M_f$ for every firm f, and
- $\mu(w) = \{f\}$ iff $w \in \mu(f)$, for every worker w and firm f.

[7] As discussed in the introduction, this example mimics Farrell and Scotchmer's (1988) study of partnerships that share the surplus equally among their members.

[8] The formal definition is presented in Section 3.

Let us use the term coalition to refer to a firm f and all workers matched to f in some matching, or to refer to an unemployed worker. Thus, a coalition may consist of a firm f and any subset of workers $S \subseteq W$ of size $\#S \leq M_f$ (including $S = \emptyset$) or of an unemployed worker. Let us denote the set of all coalitions by $\mathcal{C}$. Thus,

$$\mathcal{C} = \{\{f\} \cup S : f \in F, S \subseteq W, \#S \leq M_f\} \cup \{\{w\} : w \in W\}.$$

Note that there is a one-to-one correspondence between matchings and partitions of I into coalitions. In particular, in any matching each agent is associated with exactly one coalition.

Each agent $i \in I$ has a preference relation $\precsim_i$ over all coalitions that contain i. The profile of preferences $(\precsim_i)_{i \in I}$ is denoted by $\precsim_I$. This formulation embodies the standard assumption that each agent's preferences between two matchings are fully determined by members of the coalitions containing this agent in the two matchings.

We are interested in the existence of stable matchings in the above environment. The role of stability – most notably in preventing the unravelling of markets – has been elucidated in the empirical work started by Roth (1984). In the following definitions of pairwise stability and group stability, $C^{\mu}(i)$ denotes the coalition containing an agent i in matching μ. Specifically, the coalition containing a firm f is $C^{\mu}(f) = \{f\} \cup \mu(f)$, and the coalition containing a worker w is $C^{\mu}(w) = \mu(w) \cup \mu(\mu(w))$.

Definition 3.1 (Pairwise Stability).[9] A matching μ is blocked by a firm f if there exists a subset of workers $S \subsetneq \mu(f)$ such that $\{f\} \cup S \succ_f C^{\mu}(f)$.

A matching μ is blocked by a worker w if $\{w\} \succ_w C^{\mu}(w)$.

A matching μ is blocked by firm f and worker $w \notin \mu(f)$ if there exists $S \subseteq \mu(f)$ such that

- $\#(\{w\} \cup S) \leq M_f$,
- $\{f\} \cup \{w\} \cup S \succ_f C^{\mu}(f)$, and
- $\{f\} \cup \{w\} \cup S \succ_w C^{\mu}(w)$.

A matching is **pairwise stable** if it is not blocked by any individual agent or any worker-firm pair.

[9]Cf. Roth and Sotomayor (1990) Definition 5.3.

Definition 3.2 (Group Stability).[10] A matching μ is blocked by a group of workers and firms if there exists another matching μ' and a group A consisting of multiple workers and/or firms, such that for all workers w in A and for all firms f in A,

- $\mu'(w) \in A$ (i.e., every worker in A is matched to a firm in A);
- $C^{\mu'}(w) \succ_w C^{\mu}(w)$ (i.e., every worker in A prefers the new matching to the old one);
- $\omega \in \mu'(f)$ implies $\omega \in A \cup \mu(f)$ (i.e., every firm in A is matched to new workers only from A, although it may continue to be matched to some of its "old" workers from $\mu(f)$); and
- $C^{\mu'}(f) \succ_f C^{\mu}(f)$ (i.e., every firm in A prefers its new set of workers to its old one).

A matching is **group stable** if it is not blocked by any group of agents.

The stability concepts presuppose that a match is between a worker and a firm. Both the firm and the worker can unilaterally sever the match, and together they can establish the match irrespective of other agents' preferences.[11]

4. Division Rules and the Stability of Matching

This section answers the question posed by Section 2: what property of the equal division rule is responsible for the existence of a stable matching. As in the equal division example, the matching problems studied in this section are parametrized by a value function $\mathbf{v} : \mathcal{C} \to R_+$ that assigns value $\mathbf{v}(C)$ to each coalition $C \in \mathcal{C}$. A division rule is defined to be a function

$$D : \{(i, C, v) : i \in C \in \mathcal{C}, v \in R_+\} \to R_+$$

[10] Cf. Roth and Sotomayor (1990) Definition 5.4.

[11] In particular, even though the worker and the firm are members of a coalition composed of the firm and all its employees, other coalition members – i.e., other workers – have no veto power over the creation or severance of the firm-worker match.

that determines the allotment $D(i, C, v)$ of agent i in coalition C with coalitional value v. Given a value function $\mathbf{v}$, an agent i prefers a coalition $C \ni i$ to $C' \ni i$ iff $D(i, C, \mathbf{v}(C)) \geq D(i, C', \mathbf{v}(C'))$.

The equal division sets $D(i, C, v) = \frac{v}{\#C}$ and is an example of a division rule. In general, a division rule may represent more complex division games or bargaining protocols as illustrated in the following two examples.

Example 4.1. (Post-matching pure Nash bargaining). On date 1, firms and workers form coalitions. On this date, firms and workers cannot enter binding contracts determining the terms of employment. In effect, on date 1, the agents' preferences over coalitions reflect the agents' expectations of the payoffs that will be determined on date 2. On date 2, each resultant coalition, C, creates value $\mathbf{v}(C) \geq 0$, and its members divide $v = \mathbf{v}(C)$ into allotments s_i^{NB}. The value of an allotment s_i to agent i is $U_i(s_i)$ where an increasing and concave function U_i is i's utility or profit function. The allotments s_i are determined according to the Nash bargaining solution, that is they are set to maximize

$$\max_{s_i \geq 0} \prod_{i \in C} \left(U_i(s_i) - U_i(0)\right)$$

subject to

$$\sum_{i \in C} s_i \leq v.$$

In this example, the division rule is $D(i, C, v) = s_i^{NB}$.

Example 4.2. (Post-matching Kalai-Smorodinsky bargaining). On date 1, firms and workers form coalitions. On this date, firms and workers cannot enter binding contracts determining the terms of employment. In effect, on date 1, the agents' preferences over coalitions reflect the agents' expectations of the payoffs that will be determined on date 2. On date 2, each resultant coalition, C, creates value $\mathbf{v}(C) \geq 0$, and its members divide $v = \mathbf{v}(C)$ into allotments s_i^{KSB}. The value of an allotment s_i to agent i is $U_i(s_i)$ where an increasing and concave function U_i is i's utility or profit function. The allotments s_i are determined according to the Kalai-Smorodinsky bargaining solution, that is

$$s_i^{KSB} = \frac{U_i(v)}{\sum_{j \in C} U_j(v)} v.$$

The division rule is $D(i, C, v) = s_i^{KSB}$.

Both of these examples include situations with complementarities between workers as no assumption is made about the value function $\mathbf{v}$. The peer effects are inherent to both Nash and Kalai-Smorodinsky bargaining. Other examples – such as post-matching Tullock's (1980) rent-seeking game or linear sharing rules – are discussed in Section 6.

Note the correspondence between agents' preferences (used to define stability in section 3) and the setup with division rule and value function. Every division rule and value function determine a preference profile and every preference profile may be interpreted in terms of agents dividing coalitional values via a division rule. The division rule and value function setup will allow us to focus on the connection between stability and the institutions generating agents' payoffs.

We discusses division rules that satisfy the following regularity conditions.

Definition 4.3 (Regularity). A division rule D is *regular* if for any agent i and coalition $C \ni i$

- D has *full range*: $\{D(i, C, v) : v \geq 0\} = [0, \infty)$.
- D is *monotonic:* $D(i, C, v)$ is strictly increasing in $v \geq 0$.
- D is *continuous*: $D(i, C, v)$ is continuous in $v \geq 0$.

The equal division rule, Nash bargaining, and Kalai-Smorodinsky bargaining are regular. The role of these assumptions is discussed at the end of this section.[12]

The main result is a sufficient and necessary condition for the existence of stable matchings for all preference profiles induced by a regular division rule. This condition builds on the notion of pairwise aligned preferences. Recall that preferences are pairwise aligned if all agents in an intersection of two coalitions prefer the same coalition of the two.

Definition 4.4 (Pairwise Alignment). Preferences are pairwise aligned if for all $i, j \in I$ and coalitions $C, C' \ni i, j$, we have

$$C \precsim_i C' \Longleftrightarrow C \precsim_j C'.$$

[12] To make the setting more concrete, one may also assume that $\sum_{i \in C} D(i, C, v) \leq v$ thus imposing a feasibility-type constraint on the allotments. Such an additional assumption, however, is not necessary for the results presented. In fact, the results of this section remain true if the allotments are reinterpreted as agents' payoffs.

A division rule is pairwise aligned if it induces pairwise-aligned preferences for all value functions.

In particular, then $C \sim_i C'$ iff $C \sim_j C'$, and $C \succ_i C'$ iff $C \succ_j C'$. One can readily verify that the preferences generated by the Nash bargaining solution are pairwise aligned while preferences generated by the Kalai-Smorodinsky bargaining solution need not be pairwise aligned.

The sufficient and necessary condition for stability is given by the following.

Theorem 4.5 (Sufficiency and Necessity). Suppose that there are at least two firms and all firms are able to employ at least two workers ($M_f \geq 2$ for $f \in F$). A regular division rule is pairwise aligned if, and only if, there is a group stable matching for each induced preference profile. Moreover, if there is a pairwise stable matching for each induced preference profile then the rule is pairwise aligned.

As immediate corollaries of this theorem, we obtain

Corollary 4.6. A matching problem followed by the Nash bargaining (Example 4.1) always admits a stable matching.

Corollary 4.7. A matching problem followed by the Kalai-Smorodinsky bargaining (Example 4.2) need not admit a stable matching.

The remainder of this section first proves the sufficiency part of the theorem and then comments on the proof of the necessity part. The section ends with a discussion which assumptions may be dropped and which assumptions may be relaxed.

The proof of the sufficiency part is in two steps. The first step shows that under the assumptions of the theorem there is a metaranking. The second step of the proof is to show that if there is a metaranking, then there is a group stable matching. This step is identical to the second step of the argument in Section 2, and hence is skipped.

A metaranking is defined as follows.

Definition 4.8 (Metaranking). A metaranking is a transitive relation $\preccurlyeq$ on all coalitions such that for any $i \in I$ and $C, C' \ni i$,

$$C \precsim_i C' \Longleftrightarrow C \preccurlyeq C'.$$

An example of metaranking is the per-member value of a coalition in a matching followed by the equal division of value. Pycia (2005), the working paper version of this paper, shows that the fear of ruin coefficient defined by Aumann and Kurz (1977a, 1977b)[13] is a metaranking in a matching followed by Nash bargaining.[14]

We thus reduced the proof of the sufficiency part of Theorem 4.5 to the following.

Proposition 4.9 (Existence of Metaranking). Suppose that all firms are able to employ at least two workers ($M_f \geq 2$ for $f \in F$). If a regular division rule is pairwise aligned, then for each induced preference profile there is a metaranking.[15]

Proof. Because of monotonicity, $D(a, C, v') = D(a, C, v)$ implies $v = v'$, and hence $D(b, C, v') = D(b, C, v)$. Thus, we can define the payoff translation functions

$$t_{b,a}^{C} : [0, \infty) \to [0, \infty)$$

for each coalition C and agents $a, b \in C$ by the condition

$$t_{b,a}^{C} (D(a, C, v)) = D(b, C, v), \ \ v \geq 0.$$

For any $a, b \in C \cap C'$, the pairwise alignment guarantees that $t_{b,a}^{C} = t_{b,a}^{C'}$. Hence we can omit the superscript C in the notation for the payoff translation functions. Since there is a firm able to employ two workers, so $t_{b,a}$ is defined whenever at least one of the agents a and b is a worker.

Choose an arbitrary reference worker w^* and fix the value function $\mathbf{v} : \mathcal{C} \to R_+$. Because of the full-range assumption, $t_{w^*,a}(D(a, C, \mathbf{v}(C)))$ is well defined for any agent a and coalition $C \ni a$ even when $w^* \notin C$. By pairwise consistency,

$$t_{w^*,a}(D(a, C, \mathbf{v}(C))) = t_{w^*,a'}(D(a', C, \mathbf{v}(C)))$$

for any different $a, a' \in C$. Indeed, if $w^* \in C$ then the claim follows straightforwardly from the pairwise consistency. If $w^* \notin C$, then first consider the case when a is a firm

[13] In the notation of Example 4.1., the fear of ruin coefficient is defined as $\frac{U_i(s_i) - U_i(0)}{U_i'(s_i)}$.

[14] The existence of metaranking is a strong and desirable property of a matching setting. For example, Pycia (2005), shows that if there is a metaranking, then stable matchings are obtained as Strong Nash Equilibria (cf. Aumann (1959) and Rubinstein (1980)) of a broad class of non-cooperative matching games.

[15] This proposition further implies that if allotments (and hence payoffs) are determined by a regular pairwise-aligned division rule then the stable matching is unique for generic value functions.

and a' is a worker. Since a is able to employ two workers, $\{a, a', w^*\}$ is a coalition. By the full-range assumption, there is a value function $\mathbf{v}' : \mathcal{C} \to R_+$ such that

$$D(a', C, \mathbf{v}'(C)) = D(a', \{a, a', w^*\}, \mathbf{v}'(\{a, a', w^*\})), \text{ and}$$
$$\mathbf{v}'(C) = \mathbf{v}(C).$$

Then, the pairwise alignment implies that also

$$D(a, C, \mathbf{v}'(C)) = D(a, \{a, a', w^*\}, \mathbf{v}'(\{a, a', w^*\})).$$

Since $w^* \in \{a, a', w^*\}$, we have

$$\begin{aligned} t_{w^*,a}(D(a, C, \mathbf{v}(C))) &= t_{w^*,a}(D(a, C, \mathbf{v}'(C))) \\ &= t_{w^*,a}(D(a, \{a, a', w^*\}, \mathbf{v}'(\{a, a', w^*\}))) \\ &= t_{w^*,a'}(D(a', \{a, a', w^*\}, \mathbf{v}'(\{a', a', w^*\}))) \\ &= t_{w^*,a'}(D(a', C, \mathbf{v}'(C))) \\ &= t_{w^*,a'}(D(a', C, \mathbf{v}(C))). \end{aligned}$$

In the remaining case, both a and a' are workers. Then C contains also a firm f, and by the preceding argument

$$t_{w^*,a}(D(a, C, \mathbf{v}(C))) = t_{w^*,f}(D(f, C, \mathbf{v}(C))) = t_{w^*,a'}(D(a', C, \mathbf{v}(C))).$$

Consequently,

$$\chi(C) = t_{w^*,a}(D(a, C, \mathbf{v}(C)))$$

does not depend on a if C is fixed. The monotonicity of the division rule implies that $\chi(C)$ determines a metaranking. This completes the proof.

The necessity part of Theorem 4.5 will be proved when we prove a stronger Theorem 5.13. The proof is in the appendix to Section 5, and makes two steps. A first step considers certain configurations of coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ such that there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., 3$ (we adopt the convention that subscripts are modulo 3 that is $C_{i,i+1} = C_{3,1}$ if $i = 3$ and $C_{i-1,i} = C_{3,1}$ if $i = 1$). In these configurations, if $C_{1,2} \sim_{a_2} C_{2,3}$ and $C_{2,3} \sim_{a_3} C_{3,1}$ then $C_{1,2} \sim_{a_1} C_{3,1}$. The second steps shows then this property implies pairwise alignment.

Let us finish this section with the discussion of assumptions. First notice, that for regular division rules the pairwise alignment assumption may be formally relaxed in the following way.

Remark 4.10. If a regular division rule induces preferences such that

$$C \sim_i C' \Longleftrightarrow C \sim_j C'$$

for all $i, j \in C, C' \in \mathcal{C}$, then the division rule is pairwise aligned.[16]

The regularity assumptions may be considerably relaxed. Before discussing how they are relaxed in Section 5, let us notice that even for the sufficiency results, it is not enough to assume that a single preference profile is pairwise aligned. The following situation illustrates the problem.

Example 4.11. There are three workers w_1, w_2, w_3 and three firms $f_{1,2}, f_{2,3}, f_{3,1}$. Let us adopt the convention that the subscripts are modulo 3, that is, $w_{i+1} = w_1$ if $i = 3$. Assume that only three firm-worker coalitions $\{f_{i,i+1}, w_i, w_{i+1}\}$, $i = 1, 2, 3$, create positive payoffs for their members. Let the payoffs in coalition $\{f_{i,i+1}, w_i, w_{i+1}\}$ be such that w_i obtains 2 and w_{i+1} obtains 1.

In this example, the resultant preferences of agents are pairwise aligned. At the same time, there is no group stable matching. There are stable matchings given by the partitions $\{\{f_{i,i+1}, w_i, w_{i+1}\}, \{f_{i+1,i+2}\}, \{f_{i+2,i}\}, \{w_{i+2}\}\}$, $i = 1, 2, 3$. It is easy to modify the example so that there is no stable matching. It is enough to assume that agents' payoffs in coalitions $\{f_{i+1,i+2}, w_{i+2}\}$ are negligible, but positive.

The next section relaxes Theorems 4.5 and Proposition 4.9 in several ways. First, the monotonicity and continuity assumptions, as well as the assumption that there are at least two firms, are not needed in the sufficiency part of Theorem 4.5 and Proposition 4.9 (cf. Theorems 5.2 and 5.11).[17]

[16] Proof. Fix $i, j \in I$ and $C, C' \ni i, j$. It is enough to consider the case $i \neq j$ and $C \neq C'$. Assume that the value function $\mathbf{v}$ is such that $C \precsim_i C'$ in the induced preference profile $\precsim_I$. Use the full-range assumption to find a value function $\mathbf{v}'$ such that $\mathbf{v}'(C) = \mathbf{v}(C)$ and $C \sim_i' C'$ in the induced preference profile $\precsim_I'$. By the assumption of the lemma $C \sim_j' C'$. The monotonicity of the division rule implies that $\mathbf{v}'(C') \leq \mathbf{v}(C')$ and hence $C \precsim_j C'$. This completes the proof.

[17] In addition, the full range assumption may be relaxed to require only that $\{D(i, C, v) : v \geq 0\} = \{D(i, \{i\}, v) : v \geq 0\}$.

Second, the results may be presented in terms of ordinary preference profiles. Section 5 does that and relaxes the full-range assumption. As Example 4.11 shows, some weak variant of the full-range assumption is necessary for the results to hold.

Third, Section 5 removes the restriction that all firms are able to employ at least two workers. Theorem 5.9 replaces this restriction with an assumption on one-worker firms, that is, firms that can employ at most one worker. In particular, the sufficient condition of Theorem 5.9 is satisfied, for instance, in the Gale and Shapley (1962) marriage markets and other settings that do not admit a metaranking.

5. Preference Formulation of Stability Conditions

This section presents sufficient and necessary conditions for stability in terms of agents' ordinal preferences over coalitions. Recall that Example 4.11 shows that the pairwise alignment of preferences alone does not guarantee that a stable matching exists. The present section shows that a preference profile admits a stable matching if the pairwise-alignment assumption is satisfied by the preference profile and by some related profiles of preferences. In problems of Section 4 these related profiles are generated by a division rule. In the present section we will impose the pairwise alignment restriction directly on a domain of preference profiles. This section also shows that a preference profile may be embedded into a pairwise-aligned domain of profiles if and only if the profile satisfies a relaxed metaranking condition (defined below).

To introduce our results, let us consider a simple matching problem with payoffs determined in Nash bargaining. Suppose that two firms f_1, f_2 and two workers w_1, w_2 match on date 1. On this date, they are not able to commit to terms of employment. On date 2, each coalition creates a value and divides it according to the Nash bargaining solution. A stable matching exists in this setting as shown in Corollary 4.6. The following alternative argument for the existence of stable matching in this simple matching problem illustrates the mechanics of the proof of the general sufficiency results (Theorems 5.2 and 5.9) of this section..

If a stable matching does not exist, then there would be a cycle of coalitions such that each coalition contains an agent who strictly prefers the next coalition in the cycle. For example, worker w_1 would prefer $\{f_2, w_1, w_2\}$ to $\{f_1, w_1\}$, firm f_1 would prefer $\{f_1, w_1\}$ to $\{f_1, w_2\}$, and worker w_2 would prefer $\{f_1, w_2\}$ to $\{f_2, w_1, w_2\}$.

To show that this cannot happen, let us consider an auxiliary matching situation between firms f_1, f_2 and workers w_1, w_2 in which (i) the agents still divide the values according to the Nash bargaining solution, (ii) the values created by all coalitions except for $C = \{f_1, w_1, w_2\}$ are the same as in the original matching situation, and (iii) the value created by coalition C is such that worker w_2 is indifferent between C and $\{f_2, w_1, w_2\}$. In this auxiliary situation, the preferences of agents between coalitions from the above cycle are unchanged. The preferences are pairwise aligned because they are induced by Nash bargaining. Because of the pairwise alignment of preferences between w_2 and w_1, worker w_1 would be indifferent between C and $\{f_2, w_1, w_2\}$, and hence w_1 would prefer C to $\{f_1, w_1\}$. Again, because of the pairwise alignment of preferences between w_1 and f_1, firm f_1 would prefer C to $\{f_1, w_1\}$, and hence to $\{f_1, w_2\}$. Firm f_1's strict preference for C over $\{f_1, w_2\}$ would contradict the pairwise alignment of preferences of f_1 and w_2 over coalitions C and $\{f_1, w_2\}$.

This contradiction proves that the cycle we started with cannot occur in the auxiliary situation, and hence it cannot occur in our example. So far, we have analyzed an illustrative cycle. To complete the proof and conclude that a stable matching exists, we need to show that there are no other cycles. The argument that there are no other cycles builds on the above analysis and is further developed following the statement of Theorem 5.2, and is completed in the appendix.

The role of Nash bargaining in the above heuristic argument is to ensure that there is an auxiliary situation in which the preferences are pairwise aligned, worker w_2 is indifferent between C and $\{f_2, w_1, w_2\}$, and preferences between coalitions other than C are inherited from the original preference profile. Nash bargaining may be replaced in the above example by any other full-range division rule. Thus, the argument whose main thrust is presented above may be used to prove the sufficiency part of Theorem 4.5 even if we drop the monotonicity and continuity assumptions.

In fact, the above heuristic argument requires only that the preference profile whose stability we analyze is embedded in a domain of pairwise-aligned profiles that is rich in the following sense.

Definition 5.1 (Rich Domain). A domain of preference profiles $\mathbf{R}$ is rich if for any agent $a \in I$, any coalitions $C, C' \ni a$ such that $\#C, \#C' \geq 3$, and any $\precsim_I \in \mathbf{R}$, there exists a profile $\precsim'_I \in \mathbf{R}$ such that $C \sim'_a C'$ and all agents' $\precsim'_I$ preferences between coalitions other than C are the same as in $\precsim_I$.

The domains of preference profiles generated in the examples of Sections 2 and 4 for different value functions $v : \mathcal{C} \to R_+$ are rich. Any full-range division rule induces a rich domain of preference profiles.[18] The domain of all profiles in any matching problem is also rich.

The main result of the paper is that if a preference profile belongs to a rich domain of pairwise-aligned profiles, then there exists a stable matching. This result contains the sufficiency part of Theorem 4.5.

Theorem 5.2 (Sufficiency). Suppose that all firms are able to employ at least two workers ($M_f \geq 2$ for $f \in F$). If a preference profile $\precsim_I$ belongs to a rich domain of pairwise aligned preference profiles, then $\precsim_I$ admits a matching that is group stable.

A heuristic argument for why we may expect Theorem 5.2 to be true was presented at the beginning of this section. Let us develop it using the following notion.

Definition 5.3 (Blocking Cycle). A blocking cycle of length $m \geq 2$ is a set of coalitions $C_{1,2}, C_{2,3}, ..., C_{m,1}$ such that

(a) For $i = 1, ..., m$ there exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) There exists i such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$ and $C_{i-1,i}$ or $C_{i,i+1}$ (or both) has three or more members.

The proof of the theorem has two main steps. The first step shows that there are no blocking cycles. The second step shows that if there are no blocking cycles then there exists a group stable matching. Let us first discuss, the more difficult first step, and then the easier second step.

A blocking cycle cannot have length 2. Indeed, $C_{2,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,1}$ and the pairwise alignment imply that $C_{2,1} \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,1}$. A blocking cycle cannot have length 3 when

[18] Denoting by $u_i(C)$ agent i utility from joining coalition C, and by u_I the profile of utilities of agents $i \in I$, we may express a utility counterpart of the rich domain condition as follows. For any agent $a \in I$, coalitions $C, C' \ni a$, and any utility profile u_I there exists utility profile u'_I such that $u'_a(C) = u'_a(C')$ and $u'_j\left(\tilde{C}\right) = u_j\left(\tilde{C}\right)$ for all $j \in I$ and coalitions $\tilde{C} \neq C$. A natural question one may ask is whether on any rich domain of preference profiles one may impute utilities so that the above utility counterpart of richness is satisfied. In general, the answer is no. A counterexample is presented in Pycia (2005), the working paper draft of the present results.

one of the agents a_1, a_2, a_3 is a firm. Indeed, assume that there is a cycle

$$C_{3,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}$$

and $C_{3,1}$ has three or more members. If two or three of the agents a_1, a_2, a_3 are firms, then this is the same firm, and one can use the transitivity of this firm's preferences and the pairwise-alignment assumption to show that agents a_1, a_2, a_3 are indifferent between relevant coalitions in the cycle. If exactly one of the agents a_1, a_2, a_3 is a firm, then there is a coalition $C = \{a_1, a_2, a_3\}$ and we may use a slightly modified argument from the opening of this section.

If C is different from the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then there exists a pairwise-aligned preference profile $\precsim'_I \in \mathbf{R}$ such that

$$C \sim'_{a_3} C_{3,1}$$

and

$$C_{3,1} \precsim'_{a_1} C_{1,2} \precsim'_{a_2} C_{2,3} \precsim'_{a_3} C_{3,1}$$

and such that $\precsim_I$-indifferent agents are $\precsim'_I$ indifferent. A repeated application of the pairwise-alignment property of $\precsim'_I$, shows that

- a_1 is $\precsim'_I$ indifferent between C and $C_{3,1}$, and thus prefers C to $C_{1,2}$;
- a_2 prefers C to $C_{1,2}$, and thus to $C_{2,3}$; and
- a_3 prefers C to $C_{2,3}$, and thus to $C_{3,1}$.

None of the preferences in the cycle may be strict, as otherwise a_3 would strictly prefer C to $C_{3,1}$, contrary to a_3's indifference between these two coalitions.

If C equals one of the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then we can repeat the above argument without the need to refer to the rich domain. Hence, in both cases, we find that the cycle $C_{3,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}$ is not blocking.

To show that there are no other blocking cycles requires overcoming some obstacles. The main obstacle is the lack of a single coalition containing all agents $a_1, ..., a_m$. In fact, such a coalition does not exist if two of the agents are firms. Even when the cycle has length 3 and all agents a_1, a_2, a_3 are workers, there may not exist a coalition containing all three agents if all firms are able to employ at most two workers. How to overcome this obstacle is shown in the proof presented in the appendix.[19]

The second step in the proof of Theorem 5.2 is easier. It requires us to show that the lack of blocking cycles is a sufficient condition for stability. One could show it directly. Let us take, however, a slightly longer route, in order to re-express this sufficient condition in a more informative way, and highlight the connection with the existence of metarankings. First let us define a relaxed metaranking.

Definition 5.4 (Relaxed Metaranking). A relaxed metaranking is a transitive relation $\preccurlyeq$ on all coalitions such that

(1) For each agent $i \in I$, and coalitions $C, C' \ni i$,

$$C \precsim_i C' \text{ implies } C \preccurlyeq C'.$$

(2) For each agent $i \in I$, and coalitions $C, C' \ni i$ such that at least one of C,C' has three or more members,

$$C \preccurlyeq C' \text{ implies } C \precsim_i C'.$$

Each metaranking is also a relaxed metaranking. In the marriage problem, an identity relation on coalitions is a relaxed metaranking for any preference profile of the agents. Roughly speaking, a relaxed metaranking has two properties: (i) the coalitions higher in the ranking are preferred to the coalitions lower in the ranking by all relevant agents, and (ii) if two coalitions share the same level in the ranking, then either each of the two coalitions has at most two members or all relevant agents are indifferent between the two coalitions. In the appendix we show the following.

Lemma 5.5. There exists a relaxed metaranking if and only if there are no blocking cycles.

Given the equivalence between the lack of blocking cycles and the existence of relaxed metarankings, to complete the second step in the proof of Theorem 5.2 it is enough to show the following.

[19] Theorem 5.2 is proved as a corollary of more general Theorem 5.9, which relaxes the assumption that all firms are able to employ at least two workers. The proof of Theorem 5.9 is in the appendix.

Proposition 5.6 (Sufficiency). If there exists a relaxed metaranking, then there is a group stable matching.

Proof. The theorem is true if I contains only one agent. To prove the general case by induction, let us assume that the theorem is true for any proper subset of I.

Let $\preccurlyeq$ be a relaxed metaranking. Consider the family of coalitions

$$\mathcal{C}^{\max} = \{C : \text{there does not exist coalition } C' \text{ such that } C \prec C'\},$$

which is non-empty since there is only a finite number of coalitions and $\preccurlyeq$ is transitive.

If there is $C_0 \in \mathcal{C}^{\max}$ such that $\#(C_0) \geq 3$, then notice that $C_0 \succsim_i C$ for any $i \in C_0$ and $C \ni i$. By the inductive assumption, there exists a partition $\{C_1, ..., C_k\}$ that corresponds to a group stable matching on $I - C_0$. Then $\{C_0, C_1, ..., C_k\}$ is a partition of I that determines a group stable matching.

In the remaining case, all $C \in \mathcal{C}^{\max}$ have two or fewer members. Consider a one-to-one matching between firms from F and workers from W with preferences inherited from $\precsim_I$. By Gale and Shapley's (1962) result, there exists a group stable matching in this new problem; let

$$Q = \{C'_1, ..., C'_K\}$$

be a partition of I that corresponds to such group stable matching. We can assume that $C'_1, ..., C'_k \in \mathcal{C}^{\max}$ and $C'_{k+1}, ..., C'_K \notin \mathcal{C}^{\max}$ for some $k \geq 0$. Notice that for any $C' \in \mathcal{C}^{\max}$, any agent $i \in C'$ strictly prefers C' to any $C \notin \mathcal{C}^{\max}$ containing i. Indeed, if $C' \precsim_i C$ then $C' \preccurlyeq C$, and hence $C \in \mathcal{C}^{\max}$. Thus, $k \geq 1$.

By the inductive assumption, there is a group stable many-to-one matching on $I - C'_1 - ... - C'_k$. Let

$$\{C''_1, ..., C''_m\}$$

be the corresponding partition of $I - C'_1 - ... - C'_k$.

Now, it is enough to notice that $C'_1, ..., C'_k, C''_1, ..., C''_m$ is a group stable many-to-one matching on I. Indeed, if it is not group stable then there would exist a blocking group A that includes an agent $a \in C'_i$ for some $i \in \{1, ..., k\}$. Agent i would prefer a coalition C to C'_i. There would be two options. If $C \in \mathcal{C}^{\max}$, then matching Q would not be group stable, contrary to its construction. If $C \notin \mathcal{C}^{\max}$, then $C'_i \succ_a C$ (by the same argument that we used above to show that $k \geq 1$). This strict preference would contradict the assumption that $C_i' \precsim_a C$. This completes the proof.[20]

Theorem 5.2 presumed that each firm is able to employ at least two workers. If there are firms that cannot employ more than one worker, then the pairwise alignment condition is no longer sufficient for stability,[21] as the following example demonstrates.

Example 5.7. Let $F = \{f_1, f_2\}$ and $W = \{w_1, w_2\}$. Let the firms' employment capacities equal $M_{f_1} = 1$ and $M_{f_2} = 2$. Let the preference profile $\precsim_I$ be such that

$$\begin{aligned} \{f_1, w_1\} \succ_{w_1} \{f_2, w_1, w_2\} \succ_{w_1} \{f_2, w_1\} \succ_{w_1} \{w_1\}, \\ \{f_2, w_1, w_2\} \succ_{w_2} \{f_1, w_2\} \succ_{w_2} \{f_2, w_2\} \succ_{w_2} \{w_2\}, \\ \{f_1, w_2\} \succ_{f_1} \{f_1, w_1\} \succ_{f_1} \{f_1\}, \text{ and} \\ \{f_2, w_1, w_2\} \succ_{f_2} \{f_2, w_2\} \succ_{w_2} \{f_2, w_1\} \succ_{w_2} \{f_2\}. \end{aligned}$$

In this example there does not exist a stable matching. The reason is that in a stable matching one of the coalitions $\{f_1, w_1\}, \{f_2, w_1, w_2\}, \{f_1, w_2\}$ would need to form, but

$$\{f_1, w_1\} \succ_{w_1} \{f_2, w_1, w_2\} \succ_{w_2} \{f_1, w_2\} \succ_{f_1} \{f_1, w_1\}.$$

At the same time, $\precsim_I$ is pairwise aligned and the domain of all pairwise-aligned preference profiles is rich.

Thus, in order to extend Theorem 5.2 to cases of many-to-one matching with one-worker firms, i.e., firms with employment capacity $M_f = 1$, we need an additional assumption. The assumption is based on the idea of blocking one-worker firm, i.e., a one-worker firm that belongs to a blocking-like cycle of three coalitions.

Definition 5.8 (Blocking One-Worker Firm). A firm f unable to employ more than one worker is a blocking one-worker firm if there exists workers $w, w\prime \in W$ and a

[20] In fact, this proof demonstrates that a slightly weaker condition is sufficient for group stability. This condition says that in any subset of agents either there is a coalition that is weakly preferred by all its members to all other coalitions in the subset, or there is a group of one- and two-member coalitions that are weakly preferred by all its members to any coalition not in the group. This condition is weaker than both the existence of relaxed metaranking and the Banerjee, Konishi, and Sönmez (2001) top coalition property mentioned in the introduction.

[21] One-to-one matching is an exception. If the matching is one-to-one, then all profiles are pairwise aligned and admit stable matchings.

coalition $C \ni w, w'$ such that

$$\{f, w\} \succsim_w C \succsim_{w'} \{f, w'\} \succsim_f \{f, w\},$$

with one preference strict.

Using this notion we may state the following.

Theorem 5.9 (Sufficiency). If a preference profile belongs to a rich domain of pairwise-aligned preference profiles and there are no blocking one-worker firms, then there is a matching that is group stable. Moreover, there exists a relaxed metaranking.

This result contains Theorem 5.2 because in the latter there are no one-worker firms. This strengthened result covers the Gale and Shapley marriage market in which all preference profiles are pairwise aligned and no one-worker firm can be blocking because there are no cycles of three coalitions. There are no cycles of three coalitions because there are no firms able to employ two workers.

The heuristic for Theorem 5.9 is identical to the one for Theorem 5.2. The proof is presented in the appendix.

Before presenting the main necessity result, let us discuss two results connecting pairwise alignment, relaxed metarankings, and metarankings. The first result is an observation that every preference profile that admits a relaxed metaranking may be embedded in a rich domain of pairwise aligned preference profiles.

Proposition 5.10. (a) If a preference profile admits a relaxed metaranking, then it is pairwise aligned and there are no blocking one-worker firms.

(b) The domain of profiles admitting a relaxed metaranking is rich.

The proof of (a) is straightforward. The proof of (b) is in the appendix.

The second result says when pairwise alignment on a domain of preferences implies that there exists a metaranking.

Theorem 5.11 (Existence of Metaranking). Suppose that there is a firm able to employ two or more workers and that a domain of preference profiles $\mathbf{R}$ satisfies the following condition. For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$, there exists a profile $\precsim'_I \in \mathbf{R}$ such that $C \sim'_w C'$ and all agents' $\precsim'_I$-preferences between coalitions other than C are the same as in $\precsim_I$. If preference profiles in domain $\mathbf{R}$ are pairwise aligned and are such that there are no blocking one-worker firms, then each preference profile in $\mathbf{R}$ admits a metaranking.

The proof relies on the same ideas as the proofs of Theorems 5.2 and 5.9, and is presented in the appendix. It is easy to modify the proof of Proposition 5.10 to show that the domain of preference profiles admitting a metaranking satisfies the domain condition of Theorem 5.11.

Let us finish with a necessity counterpart of our results. The assumptions are formulated using the following notion of a perturbation of preference profile that (i) keeps all preferences between coalitions except for a reference coalition C, and (ii) perturbs agents' preferences over C in a co-monotonic way.

Definition 5.12 (Monotonic C-Perturbation). Given a coalition C, we say that a preference profile $\precsim'_I$ is a monotonic C-perturbation of a profile $\precsim_I$ if:

- For any agent $j \in I$ and coalitions $C_1, C_2 \neq C$ containing j we have
$$C_1 \precsim'_j C_2 \Longleftrightarrow C_1 \precsim_j C_2.$$
- If there is $i \in C$ and $C'' \ni i$ such that $C \succsim_i C''$ and $C \prec'_i C''$, then for any $j \in I$ and $C' \ni j$, if $C \precsim_j C'$, then $C \prec'_j C'$.
- If there is $i \in C$ and $C' \ni i$ such that $C \precsim_i C'$ and $C \succ'_i C'$, then for any $j \in I$ and $C' \ni j$, if $C \succsim_j C'$ then $C \succ'_j C'$.

For instance, the domain of preferences generated by a monotonic full-range division rule contains all monotonic C-perturbations of any profile from the domain.

Theorem 5.13 (Necessity). Suppose that either there are at least two firms able to employ two or more workers each, or that there are no such firms. Suppose also that a domain of preferences $\mathbf{R}$ satisfies the following conditions:

(1) For any agent $i \in I$, coalitions $C, C' \ni i$ such that $\#C' \geq 3$, and any $\precsim_I \in \mathbf{R}$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that $C \sim'_i C'$.

(2) For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that $C \precsim'_i C'$.

(3) For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$ such that $C \sim_i C'$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that

- $C \succ'_i C'$.
- for any $j \in C$ if $C'' \succ_j C$ then $C'' \succ'_j C$.
- for any $j \in C$ if $C'' \prec_j C$ then $C'' \prec'_j C$.

Then, if all profiles from $\mathbf{R}$ admit pairwise-stable matchings, then all profiles from $\mathbf{R}$ are pairwise aligned and are such that there are no blocking one-worker firms.[22]

This theorem generalizes the necessity part of Theorem 4.5 and is proved in the appendix. The two main steps of the proof are discussed in Section 4. The final step makes use of the following.

Remark 5.14. As in Remark 4.9, if a domain of preference profiles $\mathbf{R}$ satisfies (1), and for all $i, j \in C, C' \in \mathcal{C}$,

$$C \sim_i C' \Longleftrightarrow C \sim_j C',$$

then preferences in $\mathbf{R}$ are pairwise aligned.

The next section applies the theoretical results of the paper to some examples.

6. Applications and Examples

The results of Section 4 were illustrated with stability properties of the Nash and Kalai-Smorodinsky bargaining solutions.[23] The present section discusses three further

[22] In particular, all profiles from $\mathbf{R}$ admit relaxed metaranking. One may also notice that the domain of all preference profiles that admit a relaxed metaranking satisfies the assumptions (1)-(3).

[23] The Nash bargaining example may be generalized to the asymmetric Nash bargaining model where agent i has bargaining power λ_i and the division of value v in coalition C maximizes $\prod_{i \in C} (U_i(s_i) - U_i(0))^{\lambda_i}$ over $s_i \geq 0$, $i \in C$, subject to $\sum_{i \in C} s_i \leq v$. Furthermore, when the bargaining power of a worker w is $\lambda_w = 0$, this worker becomes a wage taker indifferent to all employment options, and a stable matching still exists.

examples in which the results of Section 4 are applicable. The division rules considered are Tullock's (1980) rent-seeking game, linear sharing rules, and the maximization of welfare objective. This section also characterizes the class of regular and Pareto optimal division rules that are pairwise aligned, and hence induce the existence of stable matchings.

As in Section 2 and Examples 4.1 and 4.2, this section considers the following setting. There are two dates. On date 1, firms and workers match but do not contract. The agents' preferences are determined by their allotments on date 2. On date 2, each coalition C realizes an allotment profile from the set of feasible allotment profiles

$$V(C) = \left\{ (s_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} s_i \leq \mathbf{v}(C) \right\},$$

where $\mathbf{v}(C)$ is the value of coalition C and $\mathbf{v} : \mathcal{C} \to R_+$ is the value function. We allow the allotments s_i to represent expected payoffs from lotteries over a larger space of outcomes.

Rent-seeking. On date 2, agents in each formed coalition $C = \{a_1, ..., a_k\}$ engage in Tullock's (1980) rent-seeking game over a prize $\mathbf{v}(C)$. Each $a_i \in C$ will be able to lobby at cost c_i to capture the prize $\mathbf{v}(C)$ with probability $\frac{c_i}{c_1+...+c_k}$. Thus, if agents expand resources $c_1, ..., c_k$ then agent a_i obtains in expectation

$$\frac{c_i}{c_1 + ... + c_k} \mathbf{v}(C) - c_i.$$

The agents play the Nash equilibrium of this rent-seeking game; every agent lobbies at cost $\frac{k-1}{k^2}\mathbf{v}(C)$ and has expected payoff $\frac{\mathbf{v}(C)}{k^2}$. By Theorem 4.5, there is a stable matching in any matching problem with payoffs determined by the Tullock rent-seeking.

Linear sharing rules. On date 2, agents divide the value using a coalition-specific linear sharing rule. The share of agent i in the value created by coalition C is $k_{i,C}$. This agent obtains the allotment

$$s_i = k_{i,C} v(C).$$

The shares $k_{i,C} > 0$ are coalition-specific, $\sum_{i \in C} k_{i,C} = 1$, and $k_{i,C}$ do not depend on the realization of $v(C)$.

In this case, the pairwise-alignment requirement takes the following simple form.

Corollary 6.1 (Sufficiency). If agents divide the values using a linear sharing rule with shares $k_{i,C}$, then there exists a stable matching if

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all C, C' and $i, j \in C \cap C'$.[24]

This corollary is an immediate consequence of Theorem 4.5 because linear sharing rules with $k_{i,C} > 0$ are regular. Notice that this corollary follows from Theorem 4.5 even if there are firms that can employ only one worker. We need, then, to reinterpret each such firm as being able to employ two workers, but generating the value 0 if employing two workers.

The condition on shares $k_{i,C}$ is also necessary, in the following sense.

Corollary 6.2 (Necessity). Suppose that there are at least two firms able to employ two or more workers each. If agents divide the values using a linear sharing rule with shares $k_{i,C}$, and there exists a stable matching for all value functions $\mathbf{v} : \mathcal{C} \to R_+$, then

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all C, C' and $i, j \in C \cap C'$.

This corollary is an immediate consequence of the necessity part of Theorem 4.5.

Notice, that if agents' utilities are $U_i(s) = s^{\lambda_i}$, then the Nash bargaining of Example 4.1 will lead to the linear division of value, and the resultant sharing rule will satisfy the above condition. Corollary 6.2 implies a partial converse of this statement. If there are firms able to employ two workers, and a profile of shares $k_{i,C}$ guarantees an existence of stable matching for all $\mathbf{v} : \mathcal{C} \to R_+$ then the shares $k_{i,C}$ may be rationalized as coming from Nash bargaining.

Welfare maximization and Pareto optimal division rules. Each formed coalition C chooses an allotment profile $\left(s_i^C\right)_{i\in C} \in R_+^{\#C}$ that maximizes the Bergson-Samuelson separable welfare functional

$$\max_{\left(s_i^C\right)_{i\in C}} \sum_{i\in C} W_i(s_i).$$

[24] Banarjee, Konishi, and Sönmez (2001) showed that this class of linear sharing rules leads to non-empty core in one-sided coalition formation. Pycia (2006) constructs a slightly larger class of linear sharing rules that guarantees the non-emptiness of the core in one-sided coalition formation. Only the linear sharing rules from this larger class guarantee that the core is non-empty for all value functions v.

subject to $\sum_{i \in C} s_i \leq v(C)$. The welfare components W_i, $i \in I$, are increasing and concave. They are agent-specific, but not coalition-specific.

Lensberg's (1987) results imply that allotments $\left(s_i^C\right)_{i \in C}$ are pairwise aligned.[25] Indeed, $\chi(C) = W_i'(s_i)$, for some $i \in C$, determine a metaranking. Hence, we obtain the following.

Corollary 6.3 (Sufficiency). If payoffs are determined by the maximization of a Bergson-Samuelson separable welfare functional, then there is a stable matching.

Lensberg's (1987) results also suggest that all Pareto optimal[26] and continuous division rules that produce pairwise-aligned profiles may be interpreted as the maximization of a Bergson-Samuelson separable welfare functional. His results cannot be directly applied in the present context, both because he considers a one-sided problem[27] and because he effectively assumes the pairwise alignment of preferences for a much larger space of applications of the choice rule than is available in our context. The appendix provides a simple proof of the following many-to-one result inspired by Lensberg (1987).

Proposition 6.4. Suppose that all firms are able to employ at least two workers. Let D be a Pareto-optimal and regular division rule. If D induces pairwise-aligned preference profiles, then there exist increasing strictly concave differentiable functions $W_i : U_i \to R$ for $i \in I$ such that $W_i'(0) = +\infty$, and

$$(D(i, C, \mathbf{v}(C)))_{i \in C} = \arg\max_{s \in V(C)} \sum_{i \in C} W_i(s_i).$$

This proposition,[28] implies the following.

[25]Lensberg (1987) studies the consistency of solution concepts. the pairwise alignment of preference profiles is related to the consistency requirement as, in many environments, a consistent solution concept generates pairwise aligned preferences. The consistency of solution concepts' idea was introduced by Harsanyi (1959) in his analysis of the independence of irrelevant alternatives in Nash bargaining. Lensberg (1987,1988), Thomson (1988), Lensberg and Thomson (1989), Hart and Mas-Collel (1989), and Young (1994) analyzed consistency in the context of Nash bargaining, welfare functions, Walrasian trade, the Shapley value, and sharing rules. Thomson (2004) gives an up-to-date survey of these results.

[26]A division rule is Pareto optimal if the payoff profile in each coalition $C \in \mathcal{C}$ is Pareto optimal in $V(C)$.

[27]For instance, Lensberg assumes that any collection of agents can form a coalition, while in many-to-one matching two firms cannot form a coalition.

[28]Both in Proposition 6.4 and Corollary 6.5, it is enough to assume that agents' payoff are Pareto

Corollary 6.5 (Necessity). Suppose that there are at least two firms and that all firms are able to employ at least two workers. Let D be a Pareto-optimal and regular division rule. If D induces preference profiles that admit stable matchings, then there exist increasing strictly concave differentiable functions $W_i : U_i \to R$ for $i \in I$ such that $W_i'(0) = +\infty$, and

$$(D(i, C, \mathbf{v}(C)))_{i \in C} = \arg\max_{s \in V(C)} \sum_{i \in C} W_i(s_i).$$

7. Conclusion

The two main contributions of the present paper are an examination of the stability properties of division rules and establishing the pairwise-alignment condition for stability. This novel condition may be used to study matching with complementarities and peer effects. The paper shows that there exists a group stable (and hence pairwise stable) matching if a regular division rule generates pairwise aligned preferences. Pairwise alignment is also a necessary condition for pairwise stability (and hence group stability).

The sufficiency and necessity results allow one to determine which sharing rules, bargaining protocols, or games induce the existence of stable matchings. There is always a stable matching if agents' preferences are induced by Nash bargaining or Tullock's (1980) rent-seeking game. The paper also applies the sufficiency and necessity results to (i) characterize the class of linear sharing rules that always induce agents' preferences such that a stable matching exists, and (ii) characterize the class of regular and Pareto optimal division rules that induce the existence of stable matchings.

Looking at a division rule instead of individual preferences provides a partial resolution to Hatfield and Milgrom (2005) and Kojima and Hatfield (2007) impossibility results on stability of matching problems in which there are complementarities between workers. In a very general model without peer effects, they showed that a weak version of substitutability is the most general condition that – *imposed on preferences of each*

optimal in a subset $V'(C)$ of the quasi-linear set $V(C)$ as long as the Pareto frontier of each $V'(C)$ is continuous in the value $v(C)$. The regularity assumption may be replaced by the monotonicity and full-range assumptions as any monotonic, full-range, and Pareto-optimal division rule is continuous, and hence regular.

agent separately – guarantees the existence of a stable matching. In this connection, the present paper shows that there are stability conditions applicable to settings with complementarities when agents' payoffs are *co-determined through division rules.*

A natural direction to extend the results of the present paper would be to generalize them to the Hatfield and Milgrom (2005) model of matching with contracts. This model incorporates as special cases the college admission setting, in which agents have preferences over coalitions, the setting in which wages are determined during matching, and the ascending package auctions. Under certain conditions,[29] such an extension of the results of the present paper is possible if there are two categories of workers. The first category encompasses the workers, such as crucial researchers in a biotech R&D lab, with whom it is not possible to write contracts because of the incompleteness of the contractual environment and the inherent complexity of the relationship between the firm and these workers. These workers might provide complementary inputs to the firm production process. The second category includes workers, such as lab assistants, with whom the firm may contract but whose inputs are substitutable.

The focus of the present paper is understanding when there exist stable matchings. In consequence, the presented sufficiency and necessity results provide a step toward understanding what interventions promote stability in matching markets. For example, consider the matching between interns (or residents) and US hospitals described by Roth (1984) and Roth and Peranson (1999). This matching is organized by the Association of American Medical Colleges, the Council on Medical Education of the American Medical Association, and the American Hospital Association that act as a social planner. The medical associations want the resultant matching to be stable because, historically, the lack of stability led to the unravelling of the intern-hospital matching process. The main instrument used by the associations is the matching algorithm. However, the associations and other policy making bodies also promote stability, or its lack, through their impact on the institutionally embedded division of value between interns and hospitals.[30]

[29]This extension requires an assumption similar to the lack of blocking one-worker firms assumed in Theorem 5.9.

[30]A recent example of such an impact is the 2003 regulation by the Accreditation Council for Graduate Medical Education that limits the residents' working hours. The majority of residents surveyed by Niederee, Knudtson, Byrnes, Helmer, and Smith (2003), and Brunworth and Sindwani (2006) supported the restriction while the majority of teaching hospitals' faculty opposed it.

An earlier example of a similar intervention were the restrictions on the number of consecutive hours residents may work imposed by the Bell Commission after Libby Zion died in the New York Hospital

Understanding the impact of the division of value on stability would thus help the associations to assess the impact of their policies on the relevant matching market and medical apprenticeship system.

References

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth (2005), "The New York City High School Match," *American Economic Review*, Papers and Proceedings 95 (2), 364-367.

Aumann, Robert J., and Mordecai Kurz (1977a), "Power and Taxes," *Econometrica* 45, 1137-1161.

Aumann, Robert J., and Mordecai Kurz (1977b), "Power and Taxes in a Multicommodity Economy," *Israel Journal of Mathematics* 27, 185-234.

Banarjee, Suryapratim, Hideo Konishi, and Tayfun Sönmez (2001), "Core in a Simple Coalition Formation Game," *Social Choice and Welfare* 18, 135-153.

Bernheim, B. Douglas, Belazel Peleg, and Michael D. Whinston (1987), "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory* 42, 1-12.

Brunworth Joseph D. and Raj Sindwani (2006), "Impact of duty hour restrictions on otolaryngology training: divergent resident and faculty perspectives," *Laryngoscope* 116(7), 1127-1130.

Dutta, Bhaskar, and Jordi Massó (1997), "Stability of Matchings When Individuals Have Preferences over Colleagues," *Journal of Economic Theory* 75(2), 464-475.

Echenique, Federico and Jorge Oviedo (2004), "Core Many-to-one Matchings by Fixed Point Methods" *Journal of Economic Theory* 115(2), 358-376.

while cared for by two sleep-deprived medical residents (cf. Robins (1995)). I am grateful to Alvin Roth for this example.

In this context, this paper distinction between a division rule and the value being divided is important as the Council has more control over the institutions that divide the payoffs than over the quality of a particular match.

Echenique, Federico and Jorge Oviedo (2006), "A Theory of Stability in Many-to-Many Matching Markets," *Theoretical Economics* 1(2), 233-273.

Echenique, Federico and Mehmet Yenmez (2007), "A Solution to Matching with Preferences over Colleagues" *Games and Economic Behavior* 59(1), 46-71.

Farrell, Joseph and Suzanne Scotchmer (1988), "Partnerships," *Quarterly Journal of Economics* 103(2), 279-297.

Gale, David, and Lloyd Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69, 9-15.

Harsanyi, John C. (1959), "A bargaining model of the cooperative n-person game." In: Tucker, A. W., and R. D. Luce (eds.), *Contributions to the theory of games IV.* Princeton University Press, 325-355.

Hart Sergiu, and Andrew Mas-Collel (1989), "Potential, value, and consistency," *Econometrica* 57(3), 589-614.

Hatfield, John W., and Paul Milgrom (2005), "Matching with Contracts," *American Economic Review* 95(4), 913-935.

Haruvy, Ernan, Alvin E. Roth, and M. Utku Ünver (forthcoming), "The Dynamics of Law Clerk Matching: An Experimental and Computational Investigation of Proposals for Reform of the Market," *Journal of Economic Dynamics and Control* (forthcoming).

Kelso, Alexander S., Jr., and Vincent P. Crawford (1982), "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica* 50, 1483-1504.

Lensberg, Terje (1988), "Stability and the Nash solution," *Journal of Economic Theory* 45, 330-341.

Lensberg, Terje (1987), "Stability and collective rationality," *Econometrica* 55, 935-961.

Lensberg, Terje, and William Thomson (1989), *Axiomatic Theory of Bargaining With a Variable Population.* Cambridge University Press.

McKinney, C. Nicholas, Muriel Niederle, and Alvin E. Roth (2005), "The collapse of a medical labor clearinghouse (and why such failures are rare)," *American Economic Review* 95 (3) 878-889.

Mark J. Niederee, Jason L. Knudtson, Matthew C. Byrnes, Stephen D. Helmer, R. Stephen Smith (2003), "A Survey of Residents and Faculty Regarding Work Hour Limitations in Surgical Training Programs" *Archives of Surgery* 138, 663-671.

Ostrovsky, Michael (2005), "Stability in Supply Chain Networks," manuscript, http://faculty-gsb.stanford.edu/ostrovsky/papers/chains.pdf.

Pycia, Marek (2005), "Many-to-One Matching without Substitutability," MIT Industrial Performance Center Working Paper 008, http://web.mit.edu/ipc/publications/pdf/05-008.pdf.

Pycia, Marek (2006), "Bargaining and Coalition Formation," manuscript, http://www.econ.psu.edu/~pycia/pycia-bargaining-and-coalition-formation.pdf.

Revilla, Pablo (2004), "Many-to-one Matching when Colleagues Matter," Working Paper of Fundacion Centro de Estudios Andaluces, E2004/85, http://public.centrodeestudiosandaluces.es/pdfs/E200485.pdf.

Robins, Natalie (1995), *The Girl Who Died Twice: Every Patient's Nightmare The Libby Zion Case and the Hidden Hazards of Hospitals,* Delacorte Press, New York, NY.

Roth, Alvin E. (1984), "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy* 92, 991-1016.

Roth, Alvin E. (1985), "The College Admissions Problem is not Equivalent to the Marriage Problem," *Journal of Economic Theory* 36, 277-288.

Roth, Alvin E. (1991) "A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the U.K.," *American Economic Review* 81, 415-440.

Roth, Alvin E. and Elliott Peranson (1999), "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review* 89(4), 748-780.

Roth, Alvin E. (2002), "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," Fisher-Schultz Lecture, *Econometrica* 70(4), 1341-1378.

Roth, Alvin E. and Elliott Peranson (1999), "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review* 89(4), 748-780.

Roth, Alvin E., and Oliveira Sotomayor (1990), *Two Sided Matching*, Cambridge University Press.

Rubinstein, Ariel (1980), "Strong Perfect Equilibrium in Supergames," *International Journal of Game Theory* 9, 1-12.

Sönmez, Tayfun (1997), "Manipulation via Capacities in Two-Sided Matching Markets," *Journal of Economic Theory* 77, 197-204.

Sönmez, Tayfun (1999), "Can Pre-Arranged Matches be Avoided in Two-Sided Matching Markets?" *Journal of Economic Theory* 86, 148-156.

Sørensen, Morten (2005) "How Smart is Smart Money? An Empirical Two-Sided Matching Model of Venture Capital," *Journal of Finance* (forthcoming)

Thomson, William (1988), "A study of choice correspondences in economies with a variable number of agents," *Journal of Economic Theory* 46, 237-254.

Thomson, William (2004), "Consistency and its Converse: an Introduction", Working Paper, University of Rochester, http://www.econ.rochester.edu/Faculty/thomson/natocons.pdf.

Tullock, Gordon (1980), "Efficient Rent Seeking," Chapter 6 in *Toward a Theory of the Rent-Seeking Society.* J. Buchanan, R. Tollison, and G. Tullock (eds.), College Station, TX: Texas A&M University Press.

Young, Peyton H. (1994), *Equity in Theory and Practice.* Princeton University Press.

Appendix

Proof of Theorem 5.2. This theorem follows from Theorem 5.9 proved below.

Proof of Lemma 5.5. ($\Longrightarrow$) For an indirect proof, consider coalitions $C_{12}, C_{23}..., C_{m1}$ such that $a_i \in C_{i-1,i} \cap C_{i,i+1}$, $i \in \{1, ..., m\}$, satisfy conditions (a) and (b) of the definition of blocking cycle. By symmetry, we can assume that $\#(C_{m,1}) \geq 3$ and $C_{m,1} \prec_{a_1} C_{1,2}$. Then $C_{1,2} \preccurlyeq C_{2,3}$, $C_{2,3} \preccurlyeq C_{3,4}$, etc., and by transitivity $C_{1,2} \preccurlyeq C_{m,1}$. Thus $C_{1,2} \precsim_{a_1} C_{m,1}$, contradicting $C_{m,1} \prec_{a_1} C_{1,2}$.

($\Longleftarrow$) Define relation $\preccurlyeq$ so that $C \preccurlyeq C'$ whenever there exists a sequence of coalitions $C_{i,i+1} \in \mathcal{C}$ such that

- $C = C_{1,2}$,
- $C' = C_{m,m+1}$, and
- there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

Then $\preccurlyeq$ is transitive. It remains to verify conditions (1) and (2). To prove (1) take $C_{1,2} = C$, $C_{2,3} = C'$ and $a_1 = i$. To prove (2), assume that C or C' has three or more members, that $i \in C \cap C'$, and that $C \preccurlyeq C'$. Now, if $C \succ_i C'$, then there would exist a blocking cycle; hence $C \precsim_i C'$. This completes the proof.

Lemma 5.9.1 (for the proof of Theorem 5.9)**.** Let the profile $\precsim_I$ belong to a rich domain $\mathbf{R}$ of pairwise-aligned preference profiles. Assume that there are no blocking one-worker firms. Then there are no cycles of three coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ such that

(a) there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$,

(b) $C_{3,1} \succsim_{a_3} C_{2,3} \succsim_{a_2} C_{1,2} \succsim_{a_1} C_{3,1}$ with at least one strict preference.

Proof. For an indirect proof, assume that there are coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ and agents a_1, a_2, a_3 that satisfy conditions (a) and (b) of the lemma. Consider the following four cases

Case 1: $a_1, a_2, a_3 \in F$. Then $a_1 = a_2 = a_3$ is a firm whose preferences are circular, which is a contradiction.

Case 2: $a_1, a_2 \in F, a_3 \in W$. Then $a_1 = a_2$ and we can shorten the cycle to $m = 2$, and use the argument from the discussion in Section 5 to derive a contradiction.

Case 3: $a_3 \in F, a_1, a_2 \in W$. The case of firm a_3 able to employ two workers was discussed in Section 5. If a_3 is able to employ at most one worker, then $C_{3,1} = \{a_1, a_3\}$ and $C_{2,3} = \{a_2, a_3\}$ and a contradiction follows from the lack of blocking one-worker firms.[31]

Case 4: $a_1, a_2, a_3 \in W$. Then, either $a_i = a_{i+1}$ for some $i = 1, 2, 3$ and the pairwise alignment directly yields a contradiction, or all a_i are different and each $C_{k,k+1}$ has three members and contains a firm able to employ two workers. Take any firm $f_0 \in F$ able to employ two workers ($M_f \geq 2$); then $\{a_1, f_0, a_2\}, \{a_2, f_0, a_3\}, \{a_3, f_0, a_1\} \in \mathcal{C}$. Without loss of generality we can assume that

$$C_{1,2} \prec_{a_2} C_{2,3}. \tag{1}$$

Furthermore, we can assume that

$$C_{1,2} \sim_{a_2} \{a_2, f_0, a_3\}, \tag{2}$$

$$C_{2,3} \sim_{a_3} \{a_3, f_0, a_1\}, \tag{3}$$

$$C_{1,2} \sim_{a_1} \{a_1, f_0, a_2\}. \tag{4}$$

Indeed, take the first of these indifferences. If $C_{1,2} = \{a_2, f_0, a_3\}$ then it is true; if $C_{1,2} \neq \{a_2, f_0, a_3\}$ then use the rich domain assumption to find a preference profile in $\mathbf{R}$ such that the above indifference is true and all preferences not involving $\{a_2, f_0, a_3\}$ are preserved. The remaining two indifferences may be obtained in an analogous way.

Combining (1) and (2) gives $\{a_2, f_0, a_3\} \prec_{a_2} C_{2,3}$. By the pairwise-alignment assumption $\{a_2, f_0, a_3\} \prec_{a_3} C_{2,3}$, and hence (3) and the pairwise alignment give

$$\{a_2, f_0, a_3\} \prec_{f_0} \{a_3, f_0, a_1\}. \tag{5}$$

Moreover, we have

$$\{a_2, f_0, a_3\} \sim_{f_0} \{a_1, f_0, a_2\}, \tag{6}$$

as otherwise (2) and (4) would imply that

$$C_{1,2} \sim_{a_1} \{a_1, f_0, a_2\} \nsim_{f_0} \{a_2, f_0, a_3\} \sim_{a_2} C_{1,2}$$

contrary to what we proved in Case 3.

Finally, combining (a), (3), (5), (6), (4), and (a) we obtain

$$C_{3,1} \succsim_{a_3} C_{2,3} \sim_{a_3} \{a_3, f_0, a_1\} \succ_{f_0} \{a_2, f_0, a_3\} \sim_{f_0} \{a_1, f_0, a_2\} \sim_{a_1} C_{1,2} \succsim_{a_1} C_{3,1}.$$

[31] This is the only place in the proof that uses the lack of blocking one-worker firms.

contrary to what we proved in Case 3 for the cycle $C_{3,1} \succsim_{a_3} \{a_3, f_0\} \succsim_{f_0} \{f_0, a_1\} \succsim_{a_1} C_{3,1}$. This completes the proof.

Proof of Theorems 5.9. For an indirect proof, assume that $\precsim_I$ belongs to a rich domain $\mathbf{R}$ of pairwise-aligned preference profiles and that $\precsim_I$ does not admit a stable matching. By Proposition 5.6 and Lemma 5.5, there exists $m \geq 2$ and a blocking cycle of coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ and agents $a_1, ..., a_n$ that satisfy conditions (a) and (b) of Definition 5.3.

If $m = 2$ then the pairwise alignment yields a contradiction, and if $m = 3$ then Lemma 5.9.1 does. For an inductive argument, fix $m \geq 4$, and assume that there are no blocking cycles of strictly fewer than m coalitions.

Step 1. Notice that $a_j \neq a_{j+2}$ for all $j = 1, ..., m$. Indeed, if $a_j = a_{j+2}$, then the pairwise alignment implies that $C_{j,j+1} \precsim_{a_j} C_{j+1,j+2}$, and at least one of the cycles

$$C_{m,1} \precsim_{a_1} \cdots \precsim_{a_{j-1}} C_{j-1,j} \precsim_{a_j} C_{j+1,j+2} \precsim_{a_{j+2}} \cdots \precsim_{a_m} C_{m,1}$$

and

$$C_{m,1} \precsim_{a_1} \cdots \precsim_{a_j} C_{j,j+1} \precsim_{a_j} C_{j+2,j+3} \precsim_{a_{j+3}} \cdots \precsim_{a_m} C_{m,1}$$

satisfies (a) and (b) and is composed of $m-1$ coalitions, which is contrary to the inductive assumption.

Step 2. There exists k such that $\#C_{k,k+1} \geq 3$, and at least one of the agents a_{k+2}, a_{k+3} is a worker. By (b) there exists i such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$, and $\#C_{i,i+1} \geq 3$ or $\#C_{i-1,i} \geq 3$. If $a_{i+2}, a_{i+3} \in F$ then $a_{i+2} = a_{i+3}$, and the cycle

$$C_{m,1} \precsim_{a_1} \cdots \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} C_{i+3,i+4} \precsim_{a_{i+4}} \cdots \precsim_{a_m} C_{m,1}$$

has $m-1$ coalitions and satisfies (a) and (b).[32] This contradiction proves the claim of Step 3.

Step 3. By Step 2, we can assume that a_3 is a worker, and $\#C_{m,1} \geq 3$ or $\#C_{1,2} \geq 3$. Let us define a three-member coalition C that contains a_1 and a_3 in two cases.

- If $a_1 \in F$ then set $C = \{a_1, a_3, w\}$ for some worker $w \in C_{m,1} \cup C_{1,2}$ such that $w \neq a_3$. Such a worker w exists, and a_1 can hire at least two workers, because one of the coalitions $C_{m,1}, C_{1,2}$ has two workers.

[32] The condition (b) is satisfied because $i - 1 \neq i + 2 \bmod m$ and $i \neq i + 2 \bmod m$ for $m \geq 4$, and hence $C_{i-1,i}, C_{i,i+1}$ are in the shorter cycle.

- If $a_1 \in W$ then set $C = \{a_1, a_3, f\}$ where f is a firm that can employ two workers (such a firm exists if there exists a blocking cycle).

By Step 1, $a_1 \neq a_3$ and hence C has indeed three members.

Step 4. Assume that $C = C_{i,i+1}$, for some $i = 1, ..., m$ (the complimentary assumption is considered in Step 5). Look at $C_{1,2}, C_{2,3}, C$ and conclude from Lemma 5.9.1 that either $C_{1,2} \prec_{a_1} C$, or $C_{2,3} \succ_{a_3} C$, or $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$.

- If $C_{1,2} \prec_{a_1} C = C_{i,i+1}$ then $i \neq 1$ and the shorter cycle

$$C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1} \prec_{a_1} C_{i,i+1}$$

satisfies (a) and (b) because $C_{m,1} \precsim_{a_1} C_{1,2} \prec_{a_1} C = C_{i,i+1}$ and $\#(C) \geq 3$. This is impossible, however, by the inductive assumption.

- If $C_{2,3} \succ_{a_3} C = C_{i,i+1}$ then $i \neq 2$ and the shorter cycle

$$C_{i,i+1} \prec_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_i} C_{i,i+1}$$

satisfies (a) and (b) because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$ and $\#(C) \geq 3$. Again, this is impossible by the inductive assumption.

- If $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$ then the cycle $C, C_{3,4} ..., C_{m,1}$ is blocking contrary to the inductive assumption.

Step 5. Finally consider the case $C \neq C_{i,i+1}$ for all i. If $\#(C_{m,1}) \geq 3$ then use the rich domain assumption to find a pairwise-aligned preference profile $\precsim_I \in \mathbf{R}$ such that there are no blocking one-worker firms, and all preferences along the blocking cycle are preserved and $C \sim_{a_1} C_{m,1}$. If $\#(C_{m,1}) < 3$ then $\#(C_{1,2}) \geq 3$ and use the rich domain assumption to find a pairwise-aligned preference profile $\precsim_I \in \mathbf{R}$ such that there are no blocking one-worker firms, and all preferences along the blocking cycle are preserved and $C \sim_{a_1} C_{1,2}$. Abusing notation let us refer to the new profile as $\precsim_I$. In both cases $C_{m,1} \precsim_{a_1} C \precsim_{a_1} C_{1,2}$.

- If $C \prec_{a_3} C_{2,3}$, then consider the collection of $m - 1$ coalitions $C, C_{3,4}, C_{4,5}, ..., C_{m,1}$. This is a blocking cycle of length $m - 1$ because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$ and $\#C \geq 3$.

- If $C \succsim_{a_3} C_{2,3}$, then consider the collection of three coalitions $C_{1,2}, C_{2,3}, C$. Since $C \precsim_{a_1} C_{1,2}$, the collection C_1, C_2, C satisfies

$$C \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C.$$

  By Lemma 5.9.1 all agents are then indifferent. But then $C, C_{3,4}..., C_{m,1}$ is a blocking cycle of $m-1$ coalitions, contrary to the inductive assumption. This completes the proof.

Proof of Proposition 5.10(b). Take $i \in I$ and coalitions $C, C' \ni i$ such that $\#C, \#C' \geq 3$ and assume that a profile $\precsim_I$ admits a relaxed metaranking $\preccurlyeq$. To prove the claim it is enough to construct a preference profile $\precsim'_I$ that admits a relaxed metaranking, and such that (i) $C \sim'_i C'$, and (ii) all agents $\precsim'_I$ preferences between coalitions other than C are same as $\precsim_I$. Let $\precsim'_I$ be determined by (ii) for coalitions other than C, and let preferences between C and any other coalition C'' be determined as follows:

$$C \precsim'_j C'' \Longleftrightarrow C' \preccurlyeq C'' \text{ and } C'' \precsim'_j C \Longleftrightarrow C'' \preccurlyeq C', \text{ for } j \in C \cap C''.$$

We will show that the transitive relation $\preccurlyeq'$ identical to $\preccurlyeq$ for coalitions other than C and such that

$$C \preccurlyeq' C'' \Longleftrightarrow C' \preccurlyeq C'' \text{ and } C'' \preccurlyeq' C \Longleftrightarrow C'' \preccurlyeq C'$$

is a relaxed metaranking for preference profile $\precsim'_I$.

It is enough to verify conditions (1) and (2) defining the relaxed metaranking in case of comparisons of C and a some coalition C''.

Condition (1) is satisfied because $C \precsim'_i C''$ means that $C' \preccurlyeq C''$, and hence $C' \preccurlyeq' C''$. A similar argument works if $C'' \precsim'_i C$.

Condition (2) is satisfied for C, irrespective of whether C or C'' has three or more members. Indeed, if $C \preccurlyeq' C''$ and the claim of the implication is false, that is, $C \succ'_j C''$, then $C' \succ C''$; and thus $C \succ' C''$, which would be a contradiction. A similar argument works if $C'' \preccurlyeq' C$. This completes the proof.

Lemma 5.11.1 (for the proof of Theorem 5.11)**.** Fix preference profile $\precsim_I$. If there are no cycles of coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ for any $m \geq 2$ such that

(a) there exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$,

(b) at least one preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$,

then $\precsim_I$ admits a metaranking.

Proof. Define relation $\preccurlyeq$ so that $C \preccurlyeq C'$ whenever there exists a sequence of coalitions $C_{i,i+1} \in \mathcal{C}'$ such that

- $C = C_{1,2}$,
- $C' = C_{m,m+1}$,
- there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

This is a transitive relation on coalitions, and it is straightforward to verify that this relation is a metaranking. This completes the proof.

Proof of Theorem 5.11. For an indirect proof, assume that $\precsim_I \in \mathbf{R}$ does not admit a metaranking. By Lemma 5.11.1, there exists $m \geq 2$ and a cycle of coalitions $C_{12}, C_{23}..., C_{m,1} \in \mathcal{C}$ and agents $a_1, ..., a_m$ that satisfy conditions (a) and (b) of the lemma.

If $m = 2$ then the pairwise alignment yields a contradiction, and if $m = 3$ then Lemma 5.9.1 does. For an inductive argument, fix $m \geq 4$, and assume that there are no blocking cycles of strictly fewer than m coalitions.

Step 1. For any i at least one agent a_i or a_{i+1} is a worker as otherwise $a_i = a_{i+1}$ and the shorter cycle

$$C_{m,1} \precsim_{a_1} ... \precsim_{a_{i-1}} C_{i-1,i} \precsim_{a_i} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1}$$

satisfies (a) and (b) contrary to the inductive assumption.

Step 2. There exists a_k and a_{k+2} that are both workers. By Step 1, there exists a_i who is a worker. If now a_{i+2} is a firm, then Step 1 implies that both a_{i+1} and a_{i+3} are workers.

Step 3. By Step 2, we may assume that a_1 and a_3 are workers. Take a firm f able to employ two or more workers, and set $C = \{a_1, a_3, f\}$.

A straightforward adaptation of Steps 4 and 5 of the proof of Theorem 5.9 gives a required contradiction and completes the proof.

Lemma 5.13.1 (for the proof of Theorem 5.13)**.** Assume that a domain $\mathbf{R}$ of preference profiles satisfies the conditions (2)-(3) of Theorem 5.13 and that all profiles in

$\mathbf{R}$ admit pairwise-stable matchings. Assume that $C_{1,2},...,C_{3,1},a_1,...,a_3$ are such that $a_i \in C_{i-1,i} \cap C_{i,i+1}$ (all subscripts modulo 3), and that

(a) if $a_i \in W$ then $\{a_i\} = C_{i-1,i} \cap C_{i,i+1}$, and

(b) if $a_i \in F$ then $C_{i,i+1} = \{a_i\} \cup S \cup \{a_{i+1}\}$ for some $S \subset C_{i-1,i}$.

Then, if $C_{3,1} \sim_{a_1} C_{1,2,}$ and $C_{1,2} \sim_{a_2} C_{2,3}$, then $C_{2,3} \succsim_{a_3} C_{3,1}$.

Proof. For an indirect proof assume that there exists a cycle $C_{1,2},...,C_{3,1}$ that satisfies (a), (b), and $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \sim_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$. Using conditions (2) and (3) we will modify the preference profile and construct a profile in $\mathbf{R}$ that does not admit a pairwise-stable matching. At each step of the procedure let us continue to denote the current profile by $\precsim_I$.

Step 1. Use (3) with $C = C_{2,3}$ and $i = a_2$ to find a preference profile $\precsim_I \in \mathbf{R}$ such that $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$. Then, use (3) with $C = C_{1,2}$ and $i = a_1$ to find $\precsim_I$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Step 2. For all agents $i \in C_{1,2} \cup ... \cup C_{3,1}$, and all coalitions $C \ni i$ different from $C_{1,2}, C_{2,3}, C_{3,1}$, use (2) to find $\precsim_i \in \mathbf{R}$ such that $C \precsim_i C_{k,k+1}$ for $k = 1,...,3$. Then, use (3) to find $\precsim_I \in \mathbf{R}$ such that $C \prec_i C_{k,k+1}$ for $k = 1,...,3$ and all $i \in I$, and $C \ni i$ different from $C_{1,2}, C_{2,3}, C_{3,1}$.

Step 3. For each $i = 1,2,3$ fix a sequence of coalitions

$$C^1_{i,i+1} \subset C^2_{i,i+1} \subset ... \subset C^{\#C_{i,i+1}}_{i,i+1} = C_{i,i+1}$$

such that

- $C^1_{i,i+1} = \{f_i\}$ for some $f_i \in F$,
- $C^{k+1}_{i,i+1} = C^k_{i,i+1} \cup \left\{a^k_i\right\}$ for some $a^k_i \in W$, $k = 1,...,\#C_{i,i+1} - 1$,
- $a^{m_i}_i = a_i, a^{m_i-1}_i = a_{i+1}$.

Recursively in k, use (2) to modify the preference profile – while preserving all strict preferences of Steps 1 and 2 – so that $C \precsim_a C^k_{i,i+1}$ for any agent $a \in C^k_{i,i+1}$ and any coalition $C \ni a$ different from $C^{k+1}_{i,i+1},...,C^{\#C_{i,i+1}}_{i,i+1}$, $k = 2,...,\#C_{i,i+1}$, $i = 1,2,3$. Then, use (3) to obtain strict preferences $C \prec_a C^k_{i,i+1}$ for any agent $a \in C^k_{i,i+1}$ and any coalition $C \ni a$ different from $C^{k+1}_{i,i+1},...,C^{\#C_{i,i+1}}_{i,i+1}$, $k = 2,...,\#C_{i,i+1}$, $i = 1,2,3$. At the same time maintain the preferences $C_{3,1} \prec_{a_1} C_{1,2} \prec_{a_2} C_{2,3} \prec_{a_3} C_{3,1}$,and $C \prec_a C_{i,i+1}$for all $a \in C \cap C_{i,i+1}$.

The resultant profile of preferences belongs to $\mathbf{R}$ and does not admit a pairwise-stable matching. This completes the proof.

Lemma 5.13.2 (for the proof of Theorem 5.13)**.** Suppose that there are at least two firms able to employ two or more workers each. Let $\mathbf{R}$ be a domain of preference profiles satisfying the condition (1) of Theorem 5.13. Assume that each profile $\precsim_I \in \mathbf{R}$ satisfies the claim of Lemma 5.13.1: for every cycle $C_{1,2}, ..., C_{3,1}, a_1, ..., a_3$ such that $a_i \in C_{i-1,i} \cap C_{i,i+1}$ and the conditions (a) and (b) are true we have

$$C_{3,1} \sim_{a_1} C_{1,2,} \text{ and } C_{1,2} \sim_{a_2} C_{2,3} \text{ imply } C_{2,3} \succsim_{a_3} C_{3,1}.$$

Then, if $A, B \in \mathcal{C}$, $B \subset A$, $\#(A - B) = 1$, and $a, b \in B$, then $A \sim_a B$ implies $A \sim_b B$.

Proof. Take $A, B \in \mathcal{C}$ such that $B \subset A$, $\#(A - B) = 1$, and take $a, b \in B$. If $a = b$ then the claim is true. If $a \neq b$, then $\#B \geq 2$ and $\#A \geq 3$. Moreover, then $A \cap B$ contains a firm that can hire two or more workers. Consider three cases.

Case 1: $a, b \in W$. As there are at least two firms, there exists $c \in F - A - B$. Consider the cycle $A, \{b, c\}, \{a, c\}$. Find $\precsim_I \in \mathbf{R}$ so that $\{a, c\} \sim_a A$ and $\{b, c\} \sim_b A$ while preferences between coalitions different than $\{b, c\}, \{a, c\}$ are preserved. Let us continue using the symbol $\precsim_I$ for new profile.

Lemma 5.13.1 implies that $\{a, c\} \sim_c \{b, c\}$. Now, $B \sim_a A$ implies $B \sim_a \{a, c\}$, and Lemma 5.13.1 applied to the cycle $B, \{a, c\}, \{b, c\}$ gives $B \sim_b \{b, c\}$. Hence, $B \sim_b \{b, c\} \sim_b A$.

Case 2: $a \in F, b \in W$. Take $c \in A - B \subset W$ and $f \in F_2 - \{a\}$; f exists since there are at least two firms able to employ two or more workers each. Let

$$C = A - \{b\} = (B \cup \{c\}) - \{b\}$$

and

$$C' = \{b, c, f\}.$$

Note that $C \cap C' = \{c\}$ and $A \cap C' = \{b\}$, so condition (a) of Lemma 5.13.1 is satisfied for the cycle C, C', A and all its permutations. Moreover, firm $a \in A \cap C$, and both $A - C$ and $C - A$ are singletons or empty. Hence also condition (b) is satisfied. Thus, the claim of Lemma 5.13.1 is satisfied for the cycle C, C', A. Similarly, the claim Lemma 5.13.1 is satisfied for the cycle C, C', B. The structure of the subsequent argument resembles Case 1.

Using (1), find a preference profile $\precsim_I \in \mathbf{R}$ that preserves preferences between coalitions other than C' and such that

$$C' \sim_b A.$$

Using (1) again, find a profile $\precsim_I \in \mathbf{R}$ that preserves preferences between coalitions other than C and such that $C \sim_c C'$. Now, Lemma 5.13.1 applied to the cycle C, C', A gives

$$C \sim_a A.$$

Since $A \sim_a B$ was preserved in the above changes of the preference profile, by transitivity of agent a's preferences we have

$$B \sim_a C.$$

Furthermore, c is indifferent between C and C'. Thus, Lemma 5.13.1 applied to B, C, C' gives

$$C' \sim_b B.$$

Since b was also shown to be indifferent between C' and A, we have $B \sim_b A$ as required.

Case 3: $a \in W, b \in F$. After renaming the agents, we can assume that $a \in F, b \in W$ and $A \sim_b B$, and use an analogue of Case 2 argument. This completes the proof.

Proof of Theorem 5.13. If there are no firms able to employ two or more workers each, then all preference profiles are consistent and there are no blocking one-worker firms.

If there are at least two firms able to employ two or more workers each, then apply Lemmas 5.13.1 and 5.13.2 to show that for all $i, j \in C, C' \in \mathcal{C}$, all profiles satisfy the condition

$$C \sim_i C' \Longrightarrow C \sim_j C'.$$

Remark 5.14 then shows that all profiles are pairwise aligned. The lack of blocking one-worker firms follows directly from Lemma 5.13.1. This completes the proof.

Proof Proposition 6.4. The proof of Proposition 4.7 for regular division rules, presented in Section 4, constructs the payoff translation functions

$$t_{b,a} : [0, \infty) \to [0, \infty)$$

for any agents a, b such that one of them is a worker. Recall that for each coalition $C \ni a, b$, and any $v \geq 0$, we have

$$t_{b,a}(D(a, C, v)) = D(b, C, v).$$

By the monotonicity of division rule D, functions $t_{b,a}$ are strictly increasing. Since D generates Pareto optimal profiles, functions $t_{b,a}$ are continuous.

Choose an arbitrary reference worker w^* and define

$$\psi_a(s) = f(t_{w^*,a}(s)), \ \ s \in [0, \infty), a \in I,$$

where $f : [0, \infty) \to R$ is a decreasing function such that $f(s) \to +\infty$ as $s \to 0+$, and such that all ψ_a are right hand side integrable at 0. Notice that there exists a function f that satisfies these conditions. Indeed, the functions $t_{w^*,a}$ for $a \in I$ are all continuous, increasing, and have value 0 at 0. Take

$$t^{\min} = \min_{a \in I} \{t_{w^*,a}\}$$

and notice that it is also continuous and increasing, and has value 0 at 0. The functions ψ_a are integrable if $f \circ t^{\min}$ is. This will be so if, for example,

$$f(t) = \left[\frac{1}{(t^{\min})^{-1}(t)}\right]^2.$$

Moreover, f is decreasing (since $t^{\min}$ is increasing), and $f(s) \to +\infty$ as $s \to 0+$ (because $t^{\min}(t) \to 0$ as $t \to 0$). Notice that ψ_a are positive and strictly decreasing and define,

$$W_a(s) = \int_0^s \psi_a(\tau)\, d\tau.$$

Now, W_a are concave and increasing.

It remains to be shown that the solution to

$$\max_{\tilde{s} \in V(C)} \sum_{a \in C} W_a(\tilde{s}_a) = \sum_{a \in C} \int_0^{\tilde{s}_a} \psi_a(\tau)\, d\tau$$

where

$$V(C) = \left\{ (s_i)_{i \in C} \in R_+^{\#C} | \sum_{i \in C} s_i \leq v \right\}$$

coincides with $D(a, C, v)$. Concavity of the problem implies that there is a solution. Since the slope at 0 for each $\int_0^{\tilde{s}_a} \psi_a(\tau)\, d\tau$ is infinite, so the solution is internal. The differentiability of the objective function implies that the internal solution is given by the first order Lagrange conditions

$$\psi_a(\tilde{s}_a) = \lambda$$

and the feasibility constraint $(\tilde{s}_a)|_{a\in C} \in V(C)$. The first order condition can be rewritten as

$$t_{w^*,a}(\tilde{s}_a) = f^{-1}(\lambda)$$

or

$$\tilde{s}_a = t_{a,w^*}\left(f^{-1}(\lambda)\right).$$

If there is no worker in C, then $C = \{f\}$ for some $f \in F$ and the Pareto optimality of D yields the claim. Otherwise, fix a worker $w \in C$ and notice that for agents $a \in C$

$$D(a, C, v) = t_{a,w}(D(a, C, v)).$$

Lemma 5.9.1 from the appendix implies

$$t_{a,w^*} \circ t_{w^*,w} = t_{a,w}.$$

Hence,

$$D(a, C, v) = t_{a,w^*}(t_{w^*,w}(D(a, C, v))) = t_{a,w^*}(x)$$

for some $x \in R$ that does not depend on a.

This equation, the analogous equation for $\tilde{s}_a$ above, the monotonicity of t_{a,w^*}, the Pareto optimality of the division rule, and the feasibility constraint $(\tilde{s}_a)|_{a\in C} \in V(C)$ imply that

$$\tilde{s}_a = D(a, C, v).$$

This completes the proof.