Non-Existence Result for Matching Games with Transfers

Marek Pycia

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Pycia (2005) studies many-to-one matching without transfers in which agents' payoffs depend on coalition-level shocks. He establishes a sufficient and necessary condition for existence of stable matchings irrespective of the realization of shocks. The present note shows that the positive results of Pycia (2005) cannot be extended to the setting with transfers: if transfers are allowed then there always is a realization of shocks for which there is no pairwise stable matching.

Model. A finite set of agents I is divided into two non-empty disjoint sets, $I = F \cup W$. We will refer to agents from F as firms, and to agents from W as workers. Each worker seeks a firm, and each firm $f \in F$ seeks up to M_f workers, where $M_f \geq 1$. A matching is a partition of agents into coalitions such that a coalition may consist of a firm f and a subset of workers $S \subseteq W$ of size $\#S \leq M_f$ (including $S = \emptyset$), or of an unemployed worker. Denote the set of coalitions by

$$\mathcal{C} = \{\{f\} \cup S : f \in F, S \subseteq W, \#S \leq M_f\} \cup \{\{w\} : w \in W\}.$$

Let $C^{\mu}(i)$ denote the coalition containing an agent i in matching μ. Each coalition C in a matching is endowed with a value $v(C)$. A matching outcome is a matching μ and a profile of payoffs $\pi = (\pi_i)_{i \in I}$ (an imputation) for each agent $i \in I$ such that for each $C \in \mu$ the sum of payoffs is $v(C)$

$$\sum_{i \in C} \pi_i = v(C).$$

Note: this model is identical to the model of Fox (2007) in which each downstream firm has a quota of 1.

[1] The material of the present note was part of an early draft of Pycia (2005) (revised later as Pycia (2007)).

Definition (Pairwise Stability). A matching outcome (μ, π) is blocked by a firm f if there exists a subset of workers $S \subseteq C^{\mu}(f)$ such that $v(\{f\} \cup S) - \sum_{i \in S} \pi_i \succ \pi_f$.

A matching outcome (μ, π) is blocked by a worker w if $v\{w\} > \pi_w$.

A matching outcome (μ, π) is blocked by firm f and worker $w \notin C^{\mu}(f)$ if there exists a subset of workers $S \subseteq C^{\mu}(f)$ such that

- $\#(\{w\} \cup S) \leq M_f$,
- $v(\{f\} \cup \{w\} \cup S) - \sum_{i \in S} \pi_i > \pi_w + \pi_f$.

A matching outcome is **pairwise stable** if it is not blocked by any individual agent or any worker-firm pair.

Proposition. Assume that $\#W, \#F \geq 3$ and $M_f \geq 2$ for three different firms firms $f \in F$. Then there exists a profile of values $v(C)|_{C \in \mathcal{C}} \in \mathbb{R}_+^{\#\mathcal{C}}$ such that there is no pairwise stable matching.

Proof. Consider three workers w_1, w_2, w_3 and three firms $f_{1,2}, f_{2,3}, f_{3,1}$ with quota $M_{f_{i,i+1}} \geq 2$. Let us adopt the convention that the subscripts are modulo 3, that is, $w_{i+1} = w_1$ if $i = 3$. Take a profile of values such that coalitions $\{f_{i,i+1}, w_i, w_{i+1}\}$, $i = 1, 2, 3$, create value 100, coalitions $\{f_{i,i+1}, w_i\}$ create value 1, and the remaining coalitions create value 0. It is straightforward to check that there does not exist a pairwise stable matching outcome. Indeed, in any candidate stable matching outcome (μ, π) there would be $i = 1, 2, 3$ such that $\{f_{i,i+1}, w_i, w_{i+1}\} \in \mu$ and $\{f_{i+2,i}, w_{i+2}\} \in \mu$. Notice furthermore that $\pi_{w_i} \leq 50$ or $\pi_{w_{i+1}} \leq 50$ (as otherwise $\pi_{f_{i,i+1}} < 0$ and $f_{i,i+1}$ would block). Similarly, $\pi_{f_{i+2,i}} \leq 1$ as otherwise w_{i+2} would block. However, if $\pi_{w_i} \leq 50$, then w_i and $f_{i+2,i}$ would block. Similarly, if $\pi_{w_{i+1}} \leq 50$, then w_{i+1} and $f_{i+2,i}$ would block. QED

References

Fox, Jeremy (2007), "Estimating Matching Games with Transfers"

Pycia, Marek (2005), "Many-to-One Matching without Substitutability." M.I.T. Industrial Performance Center Working Paper, 8/2005

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