The Cost of Ordinality

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1 Introduction

School districts and many other institutions allocating objects without the use of transfers rely on mechanisms that only elicit agents' ordinal preferences. Abdulkadiroglu, Che, and Yasuda (2011a) demonstrated that mechanisms eliciting carinal preferences can do better.[1] How large is the welfare loss? The present note shows that this loss can be arbitrarily large.

2 The Main Result

Consider a finite set of objects $X = \{1, ..., |X|\}$ such that each object x is represented by a number of identical copies $|x| \geq 0$. Consider also a finite set of agents $I = \{1, ..., |I|\}$, each of whom demands at most one object copy and evaluates the outcomes in line with the expected utility theory based on her von Neumann-Morgenstein utilities.

As Hylland and Zeckhauser (1979), Abdulkadiroglu and Sonmez (1998) and Bogomolnaia and Moulin (2001), we are interested in random assignments in which each agent i obtains a probability distribution over objects $\mu(i, \cdot)$ and the distribution satisfies the feasibility constraints

$$\sum_{i \in I} \mu(i, x) \leq |x|.$$

In this environment, there is an efficiency cost (in terms of the sum of agents' utilities) of restricting attention to ordinal strategy-proof mechanism. This efficiency cost depends on the number of object copies and the profile of agents' utilities. The loss can be zero, for instance when there is a sufficient number of object copies to allocate each agent her most preferred object. As shown by Abdulkadiroglu, Che, and Yasuda (2011a), the loss can be positive. The goal of this note is to show that the loss can be arbitrarily large.

Theorem 1. *For each $K > 0$ there is an economy and a preference profile such that the welfare of the optimal allocation is at least K times larger than the welfare of any regular, symmetric, asymptotic strategy-proof, and asymptotically ordinally efficient ordinal mechanism.*

Before embarking on the proof, let us notice that the optimal allocation can be achieved for instance in the pseudomarket mechanism of Hylland and Zeckhauser (1979). Since the pseudomarket mechanism is not necessarily incentive-compatible in finite-size markets, we discuss below two simple mechanisms that achieve the K times better allocation in a Nash equilibrium.

Proof. It is sufficient to construct an appropriate example. Consider $k + l$ agents where $l = |X| - 1$ and k is much larger than l and $|X|$. Let agents $i = 1, ..., k$ rank objects as follows $1 \succ_i 2 \succ_i ... \succ_i |X|$, and have valuations $v_i^1 = 1$ and v_i^x close to 0 for objects $x = 2, ..., |X|$. For instance we can set $v_i^x = (n - x)\epsilon$ for $x = 2, ..., |X|$ and some very small $\epsilon > 0$. Let each agent $k - 1 + \ell$ for $\ell = 2, ..., l$ have valuations $v_{k-1+\ell}^1 = 1$ and $v_{k-1+\ell}^\ell = 1 - \epsilon$ and have utility from all other objects close to 0.

Let us replicate this economy with q copies of each above agent and q copies of each object. Let us also add sufficiently many copies of object $|X|$ in order to assure that the supply of copies is sufficient to serve each agent.

By the main result in Liu and Pycia (2011), for large q all regular, strategy-proof, symmetric, and efficient ordinal mechanisms give allocations close to that of Probabilistic Serial. Probabilistic Serial allocates objects so that everybody gets share $\frac{1}{k+l}$ of object 1. Since k is much larger than l, replicas of agents $1, ..., k$ get approximately shares $\frac{1}{k}$ in objects $2, ..., |X|$ and each replica of agent $k - 1 + \ell$ for $\ell = 2, ..., l$ obtains a share of object ℓ that is bounded above by $\frac{\ell-1}{k}$. The expected utility of each replica of agents $1, ..., k$ is approximately $\frac{1}{k}$, while the expected utility of replicas of agents $k - 1 + \ell$ for $\ell = 2, ..., l$ is approximately $\frac{1}{k} + \frac{\ell-1}{k} = \frac{\ell}{k}$. The total welfare is approximately $q\left(1 + \frac{\frac{1}{2}(l-1)(l+2)}{k}\right)$, and for k much larger than l, the total welfare is essentially q.

The welfare at the efficient allocation is bounded below by the welfare of the following allocation. Each replica of agent $k - 1 + \ell$ gets share 1 of object ℓ, while each replica of the remaining (identical) agents $1, ..., k$ gets share $\frac{1}{k}$ of object 1 and the efficient allocation of the remaining objects. The total welfare of this allocation is bounded below by $q(1 + l)$. Thus, this allocation is $l + 1$ times better than the above-computed best symmetric ordinal allocation. QED

2.1 Implentation

As noted above, Hylland and Zeckhauser (1979) mechanism achieves the efficient allocation. For the problem at hand, there are other simpler mechanisms that achieve the allocation constructed in the above proof, and that achieve it in Nash equilibrium. Let us look at two such mechanisms.

First, consider the following mechanism that modifies the mechanism of Hylland and Zeck-hauser (1979) by fixing prices. Endow each agent with budget of 1, and allow the agents to purchase objects at the following prices: object 1 has price k, objects $2, ..., l+1$ has price 1, and object $l+1$ has price zero. If the aggregate demands can be served by the available supply, then allocate to each agent whatever they demanded. If the aggregate demands cannot be served by the available supply, then we ration each good proportionally to agents' demands. Under complete information about the preference profile from the proof of the theorem, it is a Nash equilibrium for each agent $1, ..., k$ to demand quantity $\frac{1}{k}$ of object 1 and quantity $1 - \frac{1}{k}$ of object $l+1$, and for each agent $k-1+\ell$ for $\ell = 2, ..., l$ to demand quantity 1 of object ℓ, thus achieving the welfare-maximizing allocation.[2]

Second, consider the following mechanism that is in the spirit of the CADA mechanism of Abdulkadiroglu, Che, and Yasuda (2011b). We ask agents for their ordinal preferences as well for their top choice among "force an ordinal mechanism" and the following $m+1$ random assignments:

$x_0 = \left(\frac{1}{k}, 0, ..., 0, \frac{k-1}{k} \right)$,

$x_\ell = (0, .., 0, 1, 0..., 0)$ with 1 in position ℓ where $\ell = 2, ..., |X|$.

If all agents chose one of the options $x_0, ..., x_{|X|}$ and if it is feasible to give everyone their chosen assignment, then we do so. Otherwise, we run the probabilistic serial mechanism (or another standard mechanism, for instance, random serial dictatorship). Provided the value of the second object is sufficiently low for agents $1, ..., k$, under complete information and the preference profile from the proof of the theorem, it is a Nash equilibrium for these agents to pick x_0 as their top choice while each agent $k-1+\ell$, where $\ell = 2, ..., l$, picks outcome x_ℓ. Furthermore, for any preference profile, in any Nash equilibrium this mechanism is at least as good as the ordinal mechanism used in its construction.

3   Conclusion

This note shows that there large potential welfare gains in developing cardinal mechanisms to for no-transfer allocation in settings such as school choice.

References

[1] Abdulkadiroglu, A., Y-K. Che, and Y. Yasuda (2011a) "Resolving Conflicting Preferences in School Choice: the Boston Mechanism Reconsidered," *American Economic Review* 101, 1-14.

[2] Abdulkadiroglu, A., Y-K. Che, and Y. Yasuda (2011b) "Expanding 'Choice' in School Choice," Working Paper.

[3] Abdulkadiroglu, A., and T. Sonmez (1998). "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problem." Econometrica 66(3), 689-701.

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[2]I would like to thank Marcin Peski for suggesting this mechanism.

[4] Bogomolnaia A. and Moulin H. (2001) "A New Solution to the Random Assignment Problem," *Journal of Economic Theory* 100, 295-328.

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[6] Liu, Q. and M. Pycia (2011) "Ordinal Efficiency, Fairness, and Incentives in Large Markets," Working Paper.