Bargaining and Coalition Formation

Marek Pycia[1]
Penn State

April 10, 2006

Abstract

This paper provides a sufficient condition for the non-emptiness of the core in coalition formation such as the formation of clubs, partnerships, firms, business alliances, and jurisdictions voting on public goods. The condition is formulated for settings in which agents first form coalitions and then each coalition realizes a payoff profile from the set of available alternatives via mechanisms such as various games, bargaining, and sharing rules. In these settings, the core is non-empty for all preference profiles induced by the mechanisms if and only if the preferences of agents over proper coalitions are pairwise aligned. The agents' preferences over proper coalitions are pairwise aligned if any two agents in the intersection of any two proper coalitions prefer the same one of the two coalitions. For instance, there exists a core coalition structure if the payoffs are determined in the Tullock rent-seeking, and Nash, egalitarian, and Rawlsian bargaining solutions. In these cases, there is generically a unique core coalition structure and the grand coalition does not necessarily form. The core might be empty if the payoffs are determined by the Kalai-Smorodinsky bargaining solution or Shapley value. The paper also determines the class of linear sharing rules and regular Pareto optimal mechanisms for which there are core coalition structures.

---

[1] Penn State Department of Economics, 418 Kern Graduate Bldg., University Park 16802. Email: pycia@psu.edu. I would like to thank Bengt Holmström, Glenn Ellison, and Hannu Vartiainen, as well as Daron Acemoglu, Abhijit Banerjee, Haluk Ergin, Robert Gibbons, Sergei Izmalkov, Michael Piore, Muhamet Yildiz, William Zame, and seminar participants at MIT Economics, MIT Industrial Performance Center, and MIT Sloan. This project was supported in part by grants from the MIT Industrial Performance Center and Leon Koźmiński Academy of Entrepreneurship and Management in Warsaw.

1. Introduction

This paper studies games of coalition formation whose outcome is a coalition structure defined as a partition of the set of agents into coalitions. All the agents have preferences over the coalitions they can join.[2] The paper focuses on situations in which agents first form coalitions and then each coalition realizes a payoff profile from the set of available alternatives. Examples of such situations include the formation of clubs (cf. Buchanan 1965), partnerships (Farrell and Scotchmer 1988), firms and business alliances (Hart and Moore 1990), jurisdictions voting on public goods (Jehiel and Scotchmer 2001).[3]

A major challenge in modelling coalition formation is that the core[4] – the standard solution concept employed to study games of coalition formation – may be empty. In effect, the models of coalition formation rely on structural restrictions to ensure that the core is non-empty that is there are core coalition structures. For example, Farrell and Scotchmer (1988) assume that the value created by a partnership is divided equally among partners. Hart and Moore (1990) assume that the coalition value depends on investments made by coalition members before they formed the coalition, the investments are complimentary at the margin, the marginal return on investment is positively correlated with the total return, and the total return is divided in Shapley bargaining within the coalition.[5]

This paper addresses this challenge. It provides a sufficient and, in a certain sense, necessary condition for the existence of core coalition structures and it provides a sufficient condition for the uniqueness of the core coalition structure. The conditions are satisfied in several bargaining settings that have not previously been recognized as admitting core coalition structures.

---

[2] Bogomolnaia and Jackson (2002) and Banerjee, Konishi and Sonmez (2001) use the term hedonic games of coalition formation to refer to such games. Drèze and Greenberg (1980) use the term hedonic to refer to the dependence of an agent's utility on who else belongs to his or her coalition.

[3] Jehiel and Scotchmer (2001) assume that there is a continuum of agents, while this paper focuses on the case of a finite number of agents.

There are also coalition formation games with special structure such as the one-to-one and many-to-one matching studied by Gale and Shapley (1962) and surveyed by Roth and Sotomayor (1990).

[4] A coalition structure is in the core if there does not exist a counterfactual coalition whose members strictly prefer it to their coalitions in the coalition structure.

[5] Hart and Moore (1990) also assume that the coalitional value is superadditive in coalition members and their assets.

The main component of the proposed condition is the pairwise alignment of preferences on proper coalitions.[6] Agents' preferences are *pairwise aligned on proper coalitions* if any two agents in the intersection of any two proper coalitions prefer the same one of the two coalitions. This condition is satisfied if agents' payoffs are determined in Nash bargaining. It is also satisfied if the payoffs are determined according to some other consistent solution concepts such as the egalitarian and Rawlsian division rules, and Tullock (1980) rent-seeking game.

However, the pairwise alignment of a single preference profile does not guarantee that the core is non-empty. For instance, in a roommate problem agents match in pairs and any two agents may form a pair. Preferences are always pairwise aligned, but the existence of a stable coalition structure is not assured.[7]

Because of the problem illustrated by the above example, the main results of the paper rely on the pairwise alignment properties of the mechanism used to determine the payoffs and not only on the pairwise alignment of a single profile of payoffs. Recall that agents first form coalitions (on date 1) and then (on date 2) each coalition chooses a profile of payoffs from the set of available alternatives. On date 1, the agents cannot negotiate binding contracts. Thus, their preferences over coalitions result from their expectations of the payoffs that will be determined on date 2. On date 2, each coalition creates and divides a coalitional value, playing a game, using a bargaining solution, a division rule, or another mechanism. This mechanism determines the agents' payoffs. The mechanism is pairwise aligned on proper coalitions if the mechanism generates agents' preferences that are pairwise aligned on proper coalitions for all coalitional values. The paper imposes some mild regularity assumptions on the mechanisms studied.

The paper's main results are as follows. It is sufficient for the existence of a core coalition structure that the mechanism is pairwise aligned on proper coalitions. For any mechanism that does not satisfy this property there exists a superadditive value function for which the mechanism generates a coalition formation problem with empty core. Thus, the pairwise alignment of the mechanism is necessary for the existence of a core coalition structure for all value functions. Moreover, the core coalition structure is generically unique if the mechanism generates agents' preferences that are pairwise aligned on all coalitions for all coalitional values.

The above sufficiency and necessity results allow one to determine which sharing rules and games induce the existence of core coalition structures. For instance, Section 5 shows

---

[6] A coalition is called proper if there is an agent that does not belong to the coalition.

[7] Cf. Example 3.2.

that there is always a generically unique core coalition structure if agents' preferences are induced by Nash bargaining, egalitarian or Rawlsian solution, or Tullock's (1980) rent-seeking game. In addition, this section shows that using the Kalai-Smorodinsky bargaining solution or the Shapley value to divide the payoffs may result in empty core. Section 6 relates the pairwise alignment condition to the literature on consistency of solution concepts. Section 7 determines the class of linear sharing rules and the class of welfare maximization mechanisms that induce the existence of core coalition structures.

The idea of using pairwise alignment to study the core seems to be new. As noted above, Farrell and Scotchmer (1988) study of the formation of partnerships shows that the core is non-empty in a coalition formation game followed by an equal division of value. Banerjee, Konishi, and Sönmez (2001) notice that the equal division may be replaced by some other linear sharing rules in Farrell and Scotchmer's analysis. In a companion paper, Pycia (2005) studies stability of many-to-one matching and shows that the pairwise alignment of preferences generated by a post-matching mechanism is crucial for the existence of stable matchings.

The paper proceeds as follows. Section 2 introduces the model. Section 3 presents examples. Section 4 presents the main results. Sections 5, 6, and 7 apply the results to determine which mechanisms generate non-empty core. The last section concludes.

2. Model

There is a finite set of agents I. A coalition structure S is a partition of I. That is in a coalition structure S, each agent $a \in I$ belongs to exactly one coalition $S(a) \in \mathcal{C} = 2^I - \{\emptyset\}$. Each agent $a \in I$ has a preference relation $\precsim_a$ over all coalitions C that contain a.[8]

Each agent $i \in I$ has a preference relation $\precsim_i$ over all coalitions that contain i. The profile of preferences $(\precsim_i)_{i \in I}$ is denoted by $\precsim_I$. This formulation embodies the assumption that each agent a is indifferent between any two coalition structures with same $S(a)$.

Agents' preferences among coalitions reflect agents' payoffs obtained in a game played after the coalitions are formed. More precisely, the payoffs are determined in the following way. There are two dates. On date 1, agents form coalitions. On this date, the

---

[8] That is each agent a is indifferent between any two coalition structures with same $S(a)$.

agents cannot negotiate binding contracts. Consequently, the agents form their preferences by foreseeing what will happen on date 2. On date 2, agents in each resultant coalition play a game that determines individual payoffs.

We are interested in the non-emptiness of the core in the above environment.

Definition 2.1 (Core). A coalition structure S is blocked by a coalition C if $C \succ_a S(a)$ for all $a \in C$. A coalition structure is in the core if it is not blocked by any coalition.

Each coalition $C \neq I$ is called proper and each coalition structure $S \neq \{I\}$ is called proper or non-trivial.

3. Examples

Let us start with an example first studied in Farrell and Scotchmer's (1988) analysis of partnerships that divide surplus equally.

Example 3.1. During coalition formation, the agents from set I of agents cannot negotiate binding contracts. The agents' preferences over coalitions are determined by date 2 payoffs. At date 2, each coalition C that formed creates value $v(C) \geq 0$ and shares it equally among its members.

In this example, the core is non-empty. Indeed, to construct a core coalition structure, take a coalition $C_1 \subseteq I$ that maximizes per member value

$$\max{}_{C_1 \subseteq I} \frac{v(C_1)}{\# C_1},$$

add a coalition $C_2 \subseteq I - C_1$ that maximizes

$$\max{}_{C_2 \subseteq I - C_1} \frac{v(C_2)}{\# C_2},$$

and recursively repeat this process until all agents are assigned to a coalition. The resulting coalition structure is in the core.

A question arises what characteristics of the equal division rule lead to the non-emptiness of the core? The answer to this question provided in this paper relies on the notion of pairwise alignment of preferences. Preferences are *pairwise aligned* if for all agents i, j and coalitions $C, C' \ni i, j$, we have

$$C \succsim_i C' \Longleftrightarrow C \succsim_j C'$$

Preferences generated by the equal division rule are pairwise aligned. Section 6 studies other value sharing mechanisms that generate pairwise aligned preferences such as the Nash bargaining.

The pairwise alignment of a preference profile is not sufficient to guarantee non-emptiness of the core as illustrated by the following.

Example 3.2. Consider the coalition formation problem with three agents a_1, a_2, a_3. Assume that each agent prefers to be in a coalition with one other agent to being alone, and prefers being alone to the grand coalition. Then, the preferences are pairwise aligned. However, if

$$\begin{aligned} \{a_3, a_1\} &\succ_{a_1} \{a_1, a_2\} \\ \{a_1, a_2\} &\succ_{a_2} \{a_2, a_3\} \\ \{a_2, a_3\} &\succ_{a_3} \{a_3, a_1\} \end{aligned}$$

then the core is empty.

The next two sections will show how pairwise alignment as a property of ex post game is sufficient, and in a certain sense necessary, condition for non-emptiness of the core.

4. Mechanisms and Coalition Formation

The basic structure of the matching problems studied in this section is similar to Example 1 of Section 3. The structure is as follows. There are two dates. On date 1, agents form coalitions. On this date, the agents cannot negotiate binding contracts. Consequently, the agents form their preferences by foreseeing what will happen on date 2. On date 2, each resultant coalition C realizes a payoff profile from the set of feasible payoffs

$$\left\{ (u_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} u_i \leq v(C) \right\},$$

where $v(C)$ is the value of coalition C and $v : \mathcal{C} \to R_+$ is the value function. We allow the payoffs u_i to represent expected payoffs from lotteries over a larger space of outcomes. Coalition C realizes a feasible payoff profile by playing some game, following some bargaining protocol, or using some sharing rule. For instance, in the example of Section 3, the payoff profile was chosen via equal division. Other examples – such as Nash bargaining, Tullock's (1980) rent-seeking game, and linear sharing rules – are discussed in Section 6.

A post-matching mechanism (or, mechanism) is a game or a choice rule that players use to decide which profile of payoffs will be realized. The following definition of a post-matching mechanism identifies each such game or rule with resulting agents' payoffs because ultimately the stability properties of any matching problem are determined by these payoffs alone.

Definition 4.1 (Mechanism). A post-matching mechanism is a function G that for every coalition C and value $v(C)$ determines nonnegative payoffs $G(i, C, v(C))$ for all members $i \in C$ so that

$$\sum_{i \in C} G(i, C, v(C)) \leq v(C).$$

For example, an equal division rule operating on a coalition C with value $v(C)$ produces payoffs $G(i, C, v(C)) = \frac{v(C)}{\#C}$.

This section discusses mechanisms that are regular in the following sense

Definition 4.2 (Regularity). A mechanism G is *regular* if for any agent i and proper coalition $C \ni i$

- G has *full range*: $\{G(i, C, v(C)) : v(C) \geq 0\} = [0, \infty)$
- G is *monotonic:* $G(i, C, \tilde{v})$ is increasing in $\tilde{v} \geq 0$
- G is *continuous*: $G(i, C, \tilde{v})$ is continuous in $\tilde{v} \geq 0$

For example, the equal division rule is regular. Also, Nash bargaining, Tullock's rent-seeking, and linear sharing rules discussed in Section 6 are regular.[9]

---

[9] A mechanism that chooses payoffs $(u_i)_{i \in C}$ that maximize a welfare functional $\sum_{i \in C} W_i(u_i)$ has full range if the welfare components W_i satisfy an Inada type condition $W_i'(u) \to 0$ as $u \to \infty$. If this condition fails, the welfare maximization mechanism may fail the full range condition, for instance, if $W_1'(u)$ and $W_2'(u)$ tend to 0 as $u \to \infty$ but $W_3'(u) > 1$ for all u.

This section provides a sufficient and necessary condition for the existence of stable matchings for all preference profiles induced by a regular mechanism. These conditions build on the notion of pairwise aligned preferences. Recall that preferences are pairwise aligned on proper coalitions if all agents in an intersection of any two proper coalitions prefer the same coalition of the two.

Definition 4.3 (Pairwise Alignment). Preferences are *pairwise aligned on proper coalitions* if for all agents i, j and proper coalitions $C, C' \ni i, j$, we have

$$C \succsim_i C' \Longleftrightarrow C \succsim_j C'$$

In particular, then $C \sim_i C'$ iff $C \sim_j C'$, and $C \succ_i C'$ iff $C \succ_j C'$. Preferences generated by the equal division rule of Example 3.1 are pairwise aligned.

The sufficient and necessary condition for stability is given by the following.

Theorem 4.4 (Sufficiency and Necessity). Assume that there are at least four agents. A regular post-matching mechanism induces preference profiles that are pairwise-aligned on proper coalitions if, and only if, the core is non-empty for each induced preference profile.

We first prove the sufficiency part, then comment on the proof of the necessity part, and end this section with a discussion of which assumptions may be dropped and which assumptions may be relaxed.

The key part of the proof of the sufficiency part relies on the following result about metarankings. A metaranking is a transitive relation on a class of coalitions that, restricted to coalitions containing any particular agent, agrees with preferences of this agent. Formally,

Definition 4.5 (Metaranking). A metaranking on coalitions from family $B \subseteq 2^I - \{\emptyset\}$ is a transitive relation $\preccurlyeq$ on coalitions from B such that for any $i \in I$ and coalitions $C, C' \in B$ that contain i,

$$C \precsim_i C' \Longleftrightarrow C \preccurlyeq C'.$$

An example of a metaranking is determined by the per-head value of a coalition in a coalition formation followed by the equal division of value. The existence of a metaranking is a strong and desirable property of a coalition formation game. For instance, Pycia (2006; Chapter 1 of the thesis) shows that if there is a metaranking, then coalition structures in the core are obtained as Strong Nash Equilibria[10] of a broad class of non-cooperative coalition formation games.[11]

Proposition 4.6 (Existence of a Metaranking on Proper Coalitions). Assume there are at least four agents. If a regular post-matching mechanism induces preference profiles pairwise-aligned on proper coalitions, then for each induced preference profile there is a metaranking on proper coalitions.

Proof. Consider proper coalition C and agent $a \in C$. Because of monotonicity, $G(a, C, v'(C)) = G(a, C, v(C))$ implies $G(b, C, v'(C)) = G(b, C, v(C))$ for any values $v(C), v'(C)$. Thus, we can define the payoff translation functions $t_{b,a}^{C} : (0, \infty) \to (0, \infty)$ for each proper coalition C and agents $a, b \in C$ by the condition

$$t_{b,a}^{C}\left(G\left(a, C, \tilde{v}\right)\right) = G\left(b, C, \tilde{v}\right), \ \tilde{v} \geq 0.$$

The pairwise alignment guarantees that $t_{b,a}^{C} = t_{b,a}^{C'}$. Thus, we can refer to translation function between a and b as $t_{b,a}$.

Choose an arbitrary reference agent w^* and fix the value function $v : \mathcal{C} \to R_+$. Because of the full range assumption, $t_{w^*,a}\left(G\left(a, C, v\left(C\right)\right)\right)$ is well defined for any agent a and proper coalition $C \ni a$ even when $w^* \notin C$. By pairwise consistency,

$$t_{w^*,a}\left(G\left(a, C, v\left(C\right)\right)\right) = t_{w^*,a'}\left(G\left(a', C, v\left(C\right)\right)\right)$$

for any different $a, a' \in C$. Indeed, if $w^* \in C$ then the claim follows straightforwardly from the pairwise consistency. If $w^* \notin C$, then by full range there is a value function $v' : \mathcal{C} \to R_+$ such that

$$\begin{aligned} G\left(a', C, v'\left(C\right)\right) &= G\left(a', \left\{a, a', w^*\right\}, v'\left(\left\{a, a', w^*\right\}\right)\right), \text{ and} \\ v'\left(C\right) &= v\left(C\right). \end{aligned}$$

---

[10] Cf. Aumann (1959), Rubinstein (1980). We may alternatively use the solution concept of Coalition-Proof Nash Equilibrium of Bernheim, Peleg, Whinston (1987).

[11] Despite the attractiveness of the existence of metarankings as a property of coalition formation games, it is difficult to use metarankings as a sufficient condition for the non-emptiness of the core. The difficulty lies in constructing an index – such as the per-head value of a coalition – for each coalition formation game. Our results solve this problem by connecting the existence of metarankings with the pairwise alignment, which is readily verifiable in a variety of settings.

Then, the pairwise alignment implies that also

$$G\left(a, C, v'\left(C\right)\right) = G\left(a, \left\{a, a', w^*\right\}, v'\left(\left\{a, a', w^*\right\}\right)\right).$$

Since $w^* \in \{a, a', w^*\}$, we have

$$\begin{aligned} t_{w^*,a}\left(G\left(a, C, v\left(C\right)\right)\right) &= t_{w^*,a}\left(G\left(a, C, v'\left(C\right)\right)\right) \\ &= t_{w^*,a}\left(G\left(a, \left\{a, a', w^*\right\}, v'\left(\left\{a, a', w^*\right\}\right)\right)\right) \\ &= t_{w^*,a'}\left(G\left(a', \left\{a, a', w^*\right\}, v'\left(\left\{a', a', w^*\right\}\right)\right)\right) \\ &= t_{w^*,a'}\left(G\left(a', C, v'\left(C\right)\right)\right) \\ &= t_{w^*,a'}\left(G\left(a', C, v\left(C\right)\right)\right). \end{aligned}$$

Consequently,

$$\chi\left(C\right) = t_{w^*,a}\left(G\left(a, C, V\left(C\right)\right)\right)$$

does not depend on a if C is fixed. Monotonicity of the mechanism implies that $\chi\left(C\right)$ determines a metaranking on proper coalitions. This completes the proof.

Proof of the sufficiency part of Theorem 4.4. Let us consider an auxiliary preference profile $(\preceq_i')_{i\in I}$ such that

$$I \preceq_i' C$$

for all $i \in I$ and proper $C \ni i$, and

$$C \preceq_i' C' \Longleftrightarrow C \preceq_i C'$$

for all $i \in I$ and proper $C, C' \ni i$. Pycia (2006)[12] shows that to prove the non-emptiness of the core for the profile $(\preceq_i)_{i\in I}$ it is enough to prove it for $(\preceq_i')_{i\in I}$. By the above

---

[12] For completeness, the relevant result is fomrulated and proved in this footnote.

Proposition. If the core of $(\precsim_i)_{i\in I}$ contains a two or more elements coalition structure and $(\precsim_i')_{i\in I}$ is equivalent to $(\precsim_i)_{i\in I}$ on proper coalitions, then the core of $(\precsim_i')_{i\in I}$ is non-empty.

Proof. If $\{I\}$ is in the core of $(\precsim_i')_{i\in I}$, then the claim is true. Also, if there is a coalition structure $S \neq \{I\}$ in the core of $(\precsim_i)_{i\in I}$ such that at least one agent $(\precsim_i')_{i\in I}$ weakly prefers S to $\{I\}$, then S is in the core with regard to preferences $(\precsim_i')_{i\in I}$, and the claim is true.

Otherwise, all agents strictly prefer $\{I\}$ to any $S \neq \{I\}$ in the core of $(\precsim_i)_{i\in I}$ and $\{I\}$ is not in the core of $(\precsim_i')_{i\in I}$. Then, there exists a coalition C such that all its members strictly prefer C to I in preferences $(\precsim_i')_{i\in I}$. Take a proper coalition structure S in the core of $(\precsim_i)_{i\in I}$. Then $C \notin S$. Hence there is an agent $a \in C$ such that $C \precsim_a S\left(a\right)$. But then a strictly prefers I to $S\left(a\right)$, weakly prefers $S\left(a\right)$ to C, and strictly prefers C to I, which is a contradiction. QED construction and Proposition 4.6, there is a metaranking on all coalitions that reflects the preference profile $(\preceq'_i)_{i\in I}$. Hence, the Farrell and Scotchmer (1988) type of argument used to prove the claim of Example 3.1 shows that the core is non-empty for $(\preceq'_i)_{i\in I}$. This completes the proof.

As an inspection of the proof shows, we can drop the assumption of there being at least four agents if preferences are pairwise aligned on all coalitions that is if for all agents i, j and coalitions $C, C' \ni i, j$, we have

$$C \succsim_i C' \Longleftrightarrow C \succsim_j C'.$$

The necessity part of Theorem 4.4 is proved in the appendix. The proof relies on the following two lemmas.

Lemma 4.7. If a regular mechanism induces preferences such that

$$C \sim_i C' \Longleftrightarrow C \sim_j C'$$

for all $i, j \in I$ and proper coalitions $C, C' \ni i, j$, then preferences generated by the mechanism are pairwise aligned on proper coalitions.

Lemma 4.8. If a regular mechanism generates preference profiles with non-empty core and coalitions $C_{1,2}, C_{2,3}, C_{3,1}$, and agents a_1, a_2, a_3 are such that[13] $\{a_i\} = C_{i-1,i} \cap C_{i,i+1}$, then,

$$C_{3,1} \sim_{a_1} C_{1,2} \text{ and } C_{1,2} \sim_{a_2} C_{2,3} \Rightarrow C_{2,3} \succsim_{a_3} C_{3,1}.$$

Let us finish this section with the discussion of assumptions. First, let us recall that Example 3.2 showed that even for the sufficiency part, it is not enough to assume that a single preference profile is pairwise aligned. Second, notice that Lemma 4.7 shows that for regular mechanisms the pairwise alignment assumption may be relaxed. Third, the monotonicity and continuity assumptions are not needed in the sufficiency part of Theorem 4.4 and Proposition 4.6 and the following result is true:

Theorem 4.9 (Sufficiency for Full-Range Mechanisms).[14] Assume there are at least four agents. If a full-range post-matching mechanism induces preference profiles

---

[13] We adopt the convention that subscripts are modulo 3 that is $C_{i,i+1} = C_{3,1}$ if $i = 3$ and $C_{i-1,i} = C_{3,1}$ if $i = 1$.

[14] Proofs of Theorems 4.9, 4.10, and 4.11 are presented in the appendix.

pairwise-aligned on proper coalitions, then the core is non-empty. Moreover, there is a metaranking on proper coalitions.

Fourth, for the necessity part of the equivalence, it is enough to assume that the core is non empty for superadditive value functions. A value function $v : \mathcal{C} \to R$ is superadditive if

$$v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$$

for any disjoint $C_1, C_2 \in \mathcal{C}$.

Theorem 4.10 (Necessity for Superadditive Values). Assume there are at least four agents. If a regular post-matching mechanism induces preference profiles with non-empty core for all superadditive value functions, then the mechanism is pairwise-aligned on proper coalitions.

Finally, if we assume pairwise alignment on all coalitions, then the core coalition structure is generically unique.

Theorem 4.11 (Uniqueness). Assume there are at least four agents. If a full-range post-matching mechanism induces preference profiles pairwise-aligned on all coalitions, then the core is non-empty and for generic value function contains a unique coalition structure. Moreover, there is a metaranking on all coalitions.

5. Applications and Examples

This section analyzes several examples of coalition formation environments and uses the results of Section 4 to determine whether the core is non-empty. The mechanisms considered are Nash bargaining, Tullock's (1980) rent-seeking game, the egalitarian and Rawlsian division rules, Kalai and Smorodinsky (1975) bargaining solution, and the Shapley value.

We consider the setting of Section 4. Recall that there are two dates. On date 1, firms and workers match but do not contract. Agents' preferences are determined by their payoffs on date 2. On date 2, each coalition C realizes a payoff profile from the set of feasible payoffs

$$V(C) = \left\{ (u_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} u_i \leq v(C) \right\},$$

where $v(C)$ is the value of coalition C and $v : \mathcal{C} \to R_+$ is the value function. We allow the payoffs u_i to represent expected payoffs from lotteries over a larger space of outcomes. Coalition C realizes a payoff profile by playing some game, following some bargaining protocol, or using some sharing rule.

Nash Bargaining. On date 2, each resultant coalition, C, creates value $v(C) \geq 0$, and its members divide $v(C)$ according to the Nash bargaining solution. That is, each agent i is endowed with an increasing and concave utility function U_i, and agents' payoffs s_i maximize

$$\max_{s_i \geq 0, i \in C} \prod_{i \in C} \left(U_i(s_i) - U_i(0) \right)$$

subject to

$$\sum_{i \in C} s_i \leq v(C).$$

Thus, agents' preferences over coalitions are induced by Nash bargaining.

Corollary 5.1. If preferences during matching are induced by Nash bargaining, then there exists a stable coalition structure. The coalition structure is generically unique.

This result is a corollary of Theorem 4.11 because Nash bargaining generates pairwise aligned preferences.[15]

---

[15] Three remarks about the Nash bargaining example might be of interest. First, the Nash structure allows for the following direct proof of theorem 5.... Let us first observe that $\frac{U_i(s_i)-U_i(0)}{U_i'(s_i)}$, called the fear of ruin coefficient (see Aumann and Kurz (1977a, 1977b) and Roth (1979)), is the same for every agent in a coalition that divides value in Nash bargaining. Indeed, the Lagrange multiplier in the Nash bargaining maximization equals the inverse of the fear of ruin, $\frac{U_i'(s_i)}{U_i(s_i)-U_i(0)}$. Additionally, the larger the fear of ruin of an agent is, the more the agent gains in a given coalition. Thus, no agents would ever want to change a coalition that maximizes their fear of ruin. Therefore, the coalition with maximal fear of ruin may be treated as if its members did not participate in the matching between the remaining agents. In this way, one can recursively construct a core coalition structure. This completes the proof.

Second, the core is non-empty when preferences come from an asymmetric Nash bargaining where agent i has bargaining power λ_i and the division of value $v(C)$ in coalition C maximizes $\prod_{i \in C} (U_i(s_i) - U_i(0))^{\lambda_i}$ over $s_i \geq 0$, $i \in C$, subject to $\sum_{i \in C} s_i \leq v(C)$. In this extension, the bargaining powers λ_i are agent-specific but are not coalition-specific.

Third, the values $v(C)$ may either accrue to the entire coalition or be composed of parts that accrue to individual members. In the latter case, the existence of a stable matching relies on the assumptions that agents' utilities are quasi-linear in a numeraire, and that, after the coalitions are determined, the agents can contract. Then, $v(C)$ is the sum of values that accrue to members in an optimal contract.

Notice that the grand coalition does not necessarily form even if the value function is superadditive. Moreover, a strong bargaining power may hurt agents by making them less desirable coalition partners.

Rent-seeking. On date 2, agents in each formed coalition $C = \{a_1, ..., a_k\}$ engage in Tullock's (1980) rent-seeking game over a prize $v(C)$. Each $a_i \in C$ will be able to lobby at cost c_i to capture the prize $v(C)$ with probability $\frac{c_i}{c_1+...+c_k}$. Thus, if agents expand resources $c_1, ..., c_k$ then agent a_i obtains in expectation

$$\frac{c_i}{c_1 + ... + c_k} v(C) - c_i.$$

The agents play the Nash equilibrium of this rent-seeking game; every agent lobbies at cost $\frac{k-1}{k^2} v(C)$ and has expected payoff $\frac{v(C)}{k^2}$. Theorem 4.4 applies and there is a stable matching in any matching problem with payoffs determined by the Tullock rent-seeking.

Egalitarian bargaining solution and the Rawlsian social choice function. Let U_i be the utility of agent i from payoff s_i. The egalitarian solution is the maximal point in the set of feasible payoffs $V(C)$ where all agents have equal utility. The Rawlsian social choice function chooses a point in $V(C)$ that maximizes the utility of the worst-off agent.[16] In our setting if the agents' utilities are continuous in (monetary) payoffs then the egalitarian solution and the Rawlsian social choice function coincide. Both solutions generate pairwise aligned payoffs, and the core is non-empty.[17]

Kalai-Smorodinsky bargaining solution. Let U_i be the utility of agent i from payoff s_i. The Kalai-Smorodinsky (1975) bargaining solution selects the Pareto optimal profile of payoffs $(s_i)_{i \in C} \in V(C)$ such that

$$\frac{U_i(s_i)}{U_j(s_j)} = \frac{U_i(v(C))}{U_j(v(C))}.$$

This solution is regular if $\lim_{t \to \infty} U_i(t) = \infty$. As the example below shows, in general this solution does not satisfy pairwise alignment, and hence there exists a value function $v : \mathcal{C} \to R_+$ for which the core is empty.

Example 5.2. Consider $I = \{1, 2, 3, 4\}$ and $U_1(s) = \log(1 + s)$ and $U_2(s) = U_3(s) = U_4(s) = s$. Then preferences of agents 1 and 2 are not aligned.

---

[16] Cf. for instance Thomson and Lensberg (1989).

[17] The solutions are regular if $U_i(t) \to \infty$ when $t \to \infty$. The index $\chi(C) = U_i(G(i, C, v(C)))$ determines a metaranking.

Shapley value. On date 2, a subcoalition C' of coalition C can unilaterally achieve the value $v^C(C')$. Assume that the values v^C are superadditive and set $v^C(\emptyset) = 0$. The Shapley value of agent $i \in C$ is given by

$$s_i = \sum_{C' \subset C} \frac{(\#C')!\,(\#C - \#C' - 1)!}{(\#C)!} \left[v^c(C' \cup \{i\}) - v^C(C')\right]$$

and is regular.

If, on date 2 each proper subcoalition of C achieves the sum of its members reservation values, then the Shapley division is equivalent to Nash bargaining, and the core is non-empty.

If, however, on date 2 each proper subcoalition C' of C can achieve value

$$v^C(C') = v(C')$$

(i.e., same value that C' would achieve if formed on date 1), then – as shown by the example below – agents' preferences are not necessarily pairwise aligned, and hence there exists a superadditive value function for which the core is empty.

Example 5.3. Consider $I = \{1, 2, 3, 4\}$ and the value functions v such that

$$\begin{aligned} v(\{1,2\}) &= v(\{1,3\}) = x \\ v(\{1,2,3\}) &= v(\{1,2,3,4\}) = y > x \end{aligned}$$

where x and y are positive parameters, and $v(C) = 0$ for remaining coalitions C. If x and y are such that agent 1 is indifferent between $\{1,2\}$ and $\{1,2,3\}$ then agent 2 prefers $\{1,2\}$ over $\{1,2,3\}$, and the preferences of agents 1 and 2 are not aligned.

6. Consistency and Pairwise Alignment

Pairwise alignment of preference profiles is related to the idea of consistency of solution concepts. Consistency is a meta-requirement and the definition of consistent solution concept vary between economic environments (cf. Thomson (2004)). In case of Pareto optimal division of a value $\tilde{v}$,[18] the definition of consistency introduced by Harsanyi (1959)[19] to study Nash bargaining might be stated as follows.

Definition 6.1. A Pareto-optimal mechanism is consistent if

$$G\left(i, C', v(C) - \sum_{j \in C - C'} G(j, C, v(C))\right) = G(i, C, v(C))$$

for any $C' \subset C$ and $i \in C'$.

In many environments, a consistent solution concept generates pairwise aligned preferences.

Theorem 6.2. A Pareto-optimal monotonic mechanism is consistent if, and only if, it generates pairwise aligned profiles.

Proof. Assume that the mechanism is consistent. Let $a, b \in C \cap C'$ and a is indifferent between C and C'. Notice that it is enough to show that b is indifferent between C and C' for $C' = \{a, b\}$. Then $C' \subset C$, and by the consistency equation

$$G\left(i, C', v(C) - \sum_{j \in C - C'} G(j, C, v(C))\right) = G(i, C, v(C))$$

for $i \in C'$. Since, a is indifferent between C and C', monotonicity implies that

$$v(C') = v(C) - \sum_{j \in C - C'} G(j, C, v(C)).$$

Hence, b is indifferent between C and C'.

---

[18] A mechanism is Pareto optimal if for all values $\tilde{v}$ it generates payoffs that are Pareto optimal that is $\sum_{i \in C} G(i, C, \tilde{v}) = \tilde{v}$.

[19] The idea of consistency of solution concepts was introduced by Harsanyi (1959) in his analysis of the independence of irrelevant alternatives in Nash bargaining. In terms of our definition, he restricted the choice of C' to two-element sets; in our setting both variants of the definition give same concept as may be seen from the proof of Theorem 6.2. Lensberg (1987,1988), Thomson (1988), Lensberg and Thomson (1989), Hart and Mas-Collel (1989), and Young (1994) analyzed related notions of consistency in the context of Nash bargaining, welfare functions, Walrasian trade, the Shapley value, and sharing rules. Thomson (2004) gives an up-to-date survey of these results.

Finally, assume that the mechanism generates pairwise aligned profiles. Pareto optimality implies that the consistency equation holds true for singleton C'. Assume that consistency equation is satisfied for C' of size n and consider C' of size $n+1$. Set

$$v\left(C'\right) = v\left(C\right) - \sum_{j \in C - C'} G\left(j, C, v\left(C\right)\right).$$

Take $a \in C'$ such that $G\left(a, C', v\left(C'\right)\right) - G\left(a, C, v\left(C\right)\right)$ is maximal. By Pareto optimality,

$$G\left(a, C', v\left(C'\right)\right) - G\left(a, C, v\left(C\right)\right) \geq 0$$

Then, by monotonicity of G and the inductive assumption for $i \in C' - \{a\}$,

$$\begin{aligned}
G\left(i, C', v\left(C'\right)\right) &= G\left(i, C' - \{a\}, v\left(C'\right) - G\left(a, C', v\left(C'\right)\right)\right) \\
&= G\left(i, C' - \{a\}, v\left(C\right) - \sum_{j \in C - C'} G\left(j, C, v\left(C\right)\right) - G\left(a, C', v\left(C'\right)\right)\right) \\
&\leq G\left(i, C' - \{a\}, v\left(C\right) - \sum_{j \in C - C'} G\left(j, C, v\left(C\right)\right) - G\left(a, C, v\left(C\right)\right)\right) \\
&= G\left(i, C, v\left(C\right)\right).
\end{aligned}$$

Pairwise alignment implies that $G\left(i, C', v\left(C'\right)\right) = G\left(i, C, v\left(C\right)\right)$ for all $i \in C'$. This ends the proof.

As an immediate consequence of Theorems 4.4 and 6.2, we obtain the following.

Corollary 6.3. Suppose there are at least four agents. Assume that a mechanism that determines the payoffs in proper coalitions is regular and Pareto-optimal. The mechanism is consistent if, and only if, the core is non-empty for all value functions.

Interestingly, a historical name used to refer to consistency was "stability," cf. Lensberg and Thomson (1989). Thus, in the old terminology, the result says that a mechanism is stable (i.e., consistent) iff it generates stable (i.e., core) coalition formation problems.

7. A Characterization of Linear Sharing Rules and Pareto Optimal Mechanisms with Non-Empty Core

This section characterizes the class of linear sharing rules and the class of Pareto optimal regular mechanisms that induce pairwise aligned preference profiles, and hence non-empty core.

Linear sharing rules. On date 2, agents divide the value using a coalition-specific linear sharing rule. The share of agent i in the value created by coalition C is $k_{i,C}$. This agent obtains

$$u_i = k_{i,C} v(C).$$

The shares $k_{i,C} > 0$ are coalition-specific, $\sum_{i \in C} k_{i,C} = 1$, and $k_{i,C}$ do not depend on the realization of $v(C)$.

In this case, the pairwise-alignment requirement takes the following simple form.

Corollary 7.1 (Sufficiency). If agents divide the values using a linear sharing rule with shares $k_{i,C}$, then there exists a stable matching if

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all proper C, C' and $i, j \in C \cap C'$.

This corollary is an immediate consequence of Theorem 4.4 because linear sharing rules with $k_{i,C} > 0$ are regular.[20] Banarjee, Konishi, and Sönmez (2001) showed that a slightly smaller class of linear sharing rules leads to non-empty one-sided core in coalition formation.

The condition on shares is also necessary, in the following sense.

Corollary 7.2 (Necessity). Suppose that there are at least two firms able to employ two or more workers each. If agents divide the values using a linear sharing rule with shares $k_{i,C}$, and there exists a stable matching for all value functions $v : \mathcal{C} \to R_+$, then

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all C, C' and $i, j \in C \cap C'$.

---

[20] We can also extend the result to allow for $k_{i,C} = 0$.

This corollary is an immediate consequence of Theorem 5.12.

Notice, that if agents' utilities are $U_i(s) = s^{\lambda_i}$, then the Nash bargaining will lead to linear division of value, and the resultant sharing rule will satisfy the above condition. Corollary 6.2 implies a partial converse of this statement. If there are firms able to employ two workers, and a profile of shares $k_{i,C}$ guarantees an existence of stable matching for all $v : \mathcal{C} \to R_+$ then the shares $k_{i,C}$ may be rationalized as coming from a Nash bargaining.

Pareto optimal regular mechanisms. Consider risk-neutral agents. On date 2, the members of each formed coalition C choose a utility profile $\left(u_i^C\right)_{i \in C} \in R_+^{\#C}$ that maximizes the Bergson-Samuelson separable welfare functional

$$\max_{\left(u_i^C\right)_{i \in C}} \sum_{i \in C} W_i(u_i).$$

subject to $\sum_{i \in C} u_i \leq v(C)$. The welfare components W_i, $i \in I$, are increasing and concave. They are agent-specific, but not coalition-specific.

Lensberg's (1987) results on consistency of welfare maximization and our Theorem 6.1 imply that payoffs $\left(u_i^C\right)_{i \in C}$ are pairwise aligned.[21] Hence, we obtain the following.

Corollary 7.3 (Sufficiency). If payoffs are determined by the maximization of a Bergson-Samuelson separable welfare functional, then the core is non-empty.

Lensberg's (1987) also showed that all Pareto optimal and continuous choice rules that produce pairwise-aligned profiles may be interpreted as maximization of a Bergson-Samuelson separable welfare functional. In view of the results of Section 6, Lensberg's result implies the following[22]

Proposition 7.4 (based on Lensberg (1987))**.** Suppose that a post-matching mechanism G has full range, is monotonic, and the payoffs $\left(G(i, C, v(C))\right)_{i \in C}$ are Pareto optimal in

$$V(C) = \left\{ (u_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} u_i \leq v(C) \right\}$$

---

[21] In fact, $\chi(C) = W_i'(u_i)$, for some $i \in C$, determines a metaranking.

[22] The appendix provides a simple proof of this result.

for all value functions $v : \mathcal{C} \rightarrow R_+$. If the mechanism induces pairwise-aligned preference profiles, then there exist increasing strictly concave differentiable functions $W_i : U_i \rightarrow R$ for $i \in I$ such that $W_i' (0) = +\infty$, and

$$(G(i, C, v(C)))_{i \in C} = \arg \max_{\sum_{i \in C} u_i \in V(C)} \sum_{i \in C} W_i (u_i).$$

This proposition[23] and Theorem 4.10 imply the following.

Corollary 7.5 (Necessity). Suppose that a post-matching mechanism G has full range, is monotonic, and the payoffs $(G(i, C, v(C)))_{i \in C}$ are Pareto optimal in

$$V(C) = \left\{ (u_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} u_i \leq v(C) \right\}$$

for any $v(C)$. If the mechanism induces preference profiles with non-empty core for superadditive value functions, then there exist increasing strictly concave differentiable functions $W_i : U_i \rightarrow R$ for $i \in I$ such that $W_i' (0) = +\infty$, and

$$(G(i, C, v(C)))_{i \in C} = \arg \max_{\sum_{i \in C} u_i \in V(C)} \sum_{i \in C} W_i (u_i).$$

8. Conclusion

This paper proposes a sufficient condition for the non-emptiness of the core. The main component of this condition is the pairwise alignment of preferences. The sufficient condition is necessary for the existence of core coalition structures for all value functions. For Pareto optimal mechanisms the condition is equivalent to the consistency of the solution concept employed by agents to divide the payoffs within each proper coalition.

The sufficiency and necessity results allow one to determine which sharing rules or games induce the existence of core coalition structures. There is always a core coalition structure if agents' preferences are induced by the Nash bargaining, egalitarian or

---

[23] Both in Proposition 6.4 and Corollary 6.5, it is enough to assume that agents' payoff are Pareto optimal in a subset $V'(C)$ of the quasi-linear set $V(C)$ as long as the Pareto frontier of each $V'(C)$ is continuous in the value $v(C)$.

Rawlsian sharing rule, or Tullock's (1980) rent-seeking game. The core may be empty if agents' preferences are determined by the Kalai-Smorodinsky bargaining or Shapley value. The paper also applies the sufficiency and necessity results to (i) determine the class of linear sharing rules that always induce agents' preferences such that a stable matching exists, and (ii) characterize the class of Pareto optimal regular mechanisms that induce the existence of core coalition structures.

All results of this paper remain true for individual stability.[24] Theorem 4.11 and the positive results of sections 5 and 7 remain true for von Neumann-Morgenstern stable set[25] but other results do not. An analogous theory is true for the core in the man-woman-child problem and other multi-sided matching problems. In many-to-one matching problems weaker conditions are sufficient and necessary for the non-emptiness of the core, and yet weaker conditions are sufficient and necessary for the existence of individually stable matchings. Finally, the results might be also adapted to the roommate problem if one replaces the pairwise alignment with the property proved in Lemma 4.8.

References

Aumann, Robert J., and Mordecai Kurz (1977a), "Power and Taxes," *Econometrica* 45, 1137-1161.

Aumann, Robert J., and Mordecai Kurz (1977b), "Power and Taxes in a Multicommodity Economy," *Israel Journal of Mathematics* 27, 185-234.

Banarjee, Suryapratim, Hideo Konishi, and Tayfun Sönmez (2001), "Core in a Simple Coalition Formation Game," *Social Choice and Welfare* 18, 135-153.

Bernheim, B. Douglas, Belazel Peleg, and Michael D. Whinston (1987), "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory* 42, 1-12.

---

[24] The results remain true for both strong and weak stability. A coalition structure S is strongly individually stable if there does not exists an agent $a \in I$ and coalition $C \in S \cup \{\emptyset\}$ such that $C \cup \{a\} \succ_a S(a)$ and $C \cup \{a\} \succsim_i S(i)$ for all $i \in C$. A coalition structure S is weakly individually stable if there does not exists an agent $a \in I$ and coalition $C \in S \cup \{\emptyset\}$ such that $C \cup \{a\} \succ_j S(j)$ for all $j \in C \cup \{a\}$.

[25] A set of coalition structures Σ is vNM stable if (internal stability) each coalition structure $S \in \Sigma$ is in weak core, and (external stability) each coalition structure $S' \notin \Sigma$ is blocked by a coalition structure $S \in \Sigma$, that means there exists a coalition $C \in S$ such that $C \succ_a S'(a)$ for every $a \in C$.

Bogomolnaia, Anna, and Matthew O. Jackson (2002), "The Stability of Hedonic Coalition Structures," *Games and Economic Behavior*, 38(2), 201-230.

Buchanan, James M. (1965) "An Economic Theory of Clubs," *Economica* 32, 1-14.

Drèze, Jacques, and Joseph Greenberg (1980), "Hedonic Coalitions: Optimality and Stability," *Econometrica* 48, 987-1003.

Farrell, Joseph and Suzanne Scotchmer (1988), "Partnerships," *Quarterly Journal of Economics* 103(2), 279-297.

Gale, David, and Lloyd Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69, 9-15.

Harsanyi, John C. (1959), "A bargaining model of the cooperative n-person game." In: Tucker, A. W., and R. D. Luce (eds.), *Contributions to the theory of games IV.* Princeton University Press, 325-355.

Hart, Oliver, and John Moore (1990), "Property Rights and the Nature of the Firm," *Journal of Political Economy*, 98(6), 1119-1158.

Hart Sergiu, and Andrew Mas-Collel (1989), "Potential, value, and consistency," *Econometrica* 57(3), 589-614.

"Constitutional Rules of Exclusion in Jurisdiction Formation", P. Jehiel and S. Scotchmer, Review of Economic Studies 68, April 2001, 393-411.

Kalai, Ehud and Meir Smorodinsky (1975), "Other solutions to Nash's bargaining problem," *Econometrica*, 43, 513-518.

Jehiel, Philippe, and Suzanne Scotchmer (2001), "Constitutional Rules of Exclusion in Jurisdiction Formation," *Review of Economic Studies* 68, 393-411.

Lensberg, Terje (1988), "Stability and the Nash solution," *Journal of Economic Theory* 45, 330-341.

Lensberg, Terje (1987), "Stability and collective rationality," *Econometrica* 55, 935-961.

Pycia, Marek (2005), "Many-to-One Matching without Substitutability," Chapter 1 of the thesis.

Pycia, Marek (2006), "The Grand Coalition and the Core in Hedonic Games of Coalition Formation," manuscript.

Roth, Alvin E. (1979), *Axiomatic Models of Bargaining.* Berlin-Heidleberg-New York: Springer Verlag.

Roth, Alvin E., and Oliveira Sotomayor (1990), *Two Sided Matching*, Cambridge University Press.

Rubinstein, Ariel (1980), "Strong Perfect Equilibrium in Supergames," *International Journal of Game Theory* 9, 1-12.

Thomson, William (1988), "A study of choice correspondences in economies with a variable number of agents," *Journal of Economic Theory* 46, 237-254.

Thomson, William (2004), "Consistency and its Converse: an Introduction", Working Paper, University of Rochester

Thomson, William, and Terje Lensberg (1989), *Axiomatic Theory of Bargaining With a Variable Population.* Cambridge University Press.

Tullock, Gordon (1980), "Efficient Rent Seeking," Chapter 6 in *Toward a Theory of the Rent-Seeking Society.* J. Buchanan, R. Tollison, and G. Tullock (eds.), College Station, TX: Texas A&M University Press.

Young, Peyton H. (1994), *Equity in Theory and Practice.* Princeton University Press.

Appendices to Sections 4 and 7

Appendix to Section 4.

Proof of Lemma 4.7. Fix $i, j \in I$ and proper $C, C' \ni i, j$. It is enough to consider the case $i \neq j$ and $C \neq C'$. Assume that the value function v is such that $C \precsim_i C'$ in the induced preference profile $\precsim_I$. Use the full range assumption to find a value $v'(C)$ such that $C \sim_i' C'$ in the induced preference profile $\precsim_I'$. Then, $v'(C) \geq v(C)$ and $C \sim_j' C'$. The monotonicity of the mechanism implies that $C \precsim_j C'$. This completes the proof.

Proof of Lemma 4.8. For an indirect proof assume that there exists a cycle $C_{1,2}, ..., C_{3,1}$ that satisfies assumptions of the lemma but $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \sim_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Use monotonicity and continuity of the mechanism to find a profile of coalition values such that $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$ (we continue to denote the new preference profile by the same symbol). Repeating this argument, find a profile of values such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Lower the values on all coalitions C different from $C_{1,2}, C_{2,3}, C_{3,1}$ so that for all $i \in C_{1,2} \cup C_{2,3} \cup C_{3,1}$ we have

$$C \prec C_{i,i+1}.$$

The resultant profile of preferences does not admit a stable matching. This completes the proof.

Proof of Theorem 4.4 (necessity part). By Lemma 4.7, it is enough to take proper coalitions A, B and $a, b \in A \cap B$, and show that

$$A \sim_a B \Rightarrow A \sim_b B.$$

Furthermore, to prove this implication it is enough to show it for $B = \{a, b\}$. Thus assume that $A \sim_a B = \{a, b\}$. Let $c \in I - A \subset I - B$. Change the values on $\{a, c\}$ and $\{b, c\}$ so that

$$\begin{aligned} \{a, c\} &\sim_a A \\ \{b, c\} &\sim_c \{a, c\}. \end{aligned}$$

Then, by Lemma 4.8,

$$\{b, c\} \sim_b A.$$

Moreover, we have

$$\begin{aligned} \{a, c\} &\sim_a B \\ \{b, c\} &\sim_c \{a, c\} \end{aligned}$$

Thus, by Lemma 4.8,

$$\{b, c\} \sim_b B$$

and the proof is completed.

Definition 4.9.1 (Rich Domain). A domain of preference profiles $\mathbf{R}$ is rich if for any agent $i \in I$, proper coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$, there exists a profile $\precsim'_I \in \mathbf{R}$ such that $C \sim'_i C'$ and all agents' $\precsim'_I$ preferences between coalitions other than C are the same as in $\precsim_I$.

A domain of all preference profiles that might be generated in the equal division rule of Section 3 for different value functions $v : \mathcal{C} \to R_+$ is rich. Any full-range mechanism induces a rich domain of preference profiles when applied to different configurations of coalitions' payoff profile sets. The domain of all profiles in any coalition formation problem is also rich.

Lemma 4.9.2. Assume that there are at least four agents. Let the profile $\precsim_I$ belong to a rich domain $\mathbf{R}$ of pairwise-aligned preference profiles. Then there are no cycles of three proper coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ such that

(a) there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$,

(b) $C_{3,1} \succsim_{a_3} C_{2,3} \succsim_{a_2} C_{1,2} \succsim_{a_1} C_{3,1}$ with at least one strict preference .

Proof. For an indirect proof, assume that there are proper coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ satisfying (a) and (b). Consider $C = \{a_1, a_2, a_3\}$. If C is different from the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then there exists a pairwise-aligned preference profile $\precsim'_I \in \mathbf{R}$ such that

$$C \sim'_{a_3} C_{3,1}$$

and

$$C_{3,1} \precsim'_{a_1} C_{1,2} \precsim'_{a_2} C_{2,3} \precsim'_{a_3} C_{3,1}$$

with indifference if there was an $\precsim_I$ indifference in the cycle. A repeated application of the pairwise-alignment property of $\precsim'_I$, shows that

- a_1 is $\precsim'_I$ indifferent between C and $C_{3,1}$, and thus prefers C to $C_{1,2}$;
- a_2 prefers C to $C_{1,2}$, and thus to $C_{2,3}$; and
- a_3 prefers C to $C_{2,3}$, and thus to $C_{3,1}$.

None of the preferences on the cycle may be strict, as otherwise a_3 would strictly prefer C to $C_{3,1}$, contrary to a_3's indifference between these two coalitions.

If C equals one of the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then we can repeat the above argument without the need to refer to the rich domain. This completes the proof.

Lemma 4.9.3. There exists a metaranking on proper coalitions if and only if there is no cycle of proper coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ for some $m \geq 2$ such that

(a) There exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) For at least one i the preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

Proof. ($\Longrightarrow$) For an indirect proof, consider coalitions $C_{12}, C_{23}..., C_{m1}$ such that $a_i \in C_{i-1,i} \cap C_{i,i+1}$, $i \in \{1, ..., m\}$, satisfy conditions (a) and (b) of the definition of a blocking cycle. Let $C_{m,1} \prec_{a_1} C_{1,2}$. Then $C_{1,2} \preccurlyeq C_{2,3}$, $C_{2,3} \preccurlyeq C_{3,4}$, etc., and by transitivity $C_{1,2} \preccurlyeq C_{m,1}$. Thus $C_{1,2} \precsim_{a_1} C_{m,1}$, contradicting $C_{m,1} \prec_{a_1} C_{1,2}$.

($\Longleftarrow$) Define relation $\preccurlyeq$ so that $C \preccurlyeq C'$ whenever there exists a sequence of proper coalitions $C_{i,i+1} \in \mathcal{C}$ such that

- $C = C_{1,2}$,
- $C' = C_{m,m+1}$, and
- there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$ for certain i.

Then $\preccurlyeq$ is transitive. It remains to verify that for proper C, C' with $i \in C \cap C'$

$$C \preccurlyeq C' \Leftrightarrow C \precsim_i C'$$

To prove the first implication take $C_{1,2} = C$, $C_{2,3} = C'$ and $i = a_1$. To prove the reverse implication assume that C or C' are proper, $i \in C \cap C'$, and $C \preccurlyeq C'$. Now, if $C \succ_i C'$, then there would exist a blocking cycle; hence $C \precsim_i C'$. This completes the proof.

Proof of Theorems 4.9. For an indirect proof, assume that the core for $\precsim_I$ is empty. In particular, a metaranking on proper coalitions does not exist. By Lemma 4.9.3, the lack of a metaranking on proper coalitions means that there exists a blocking cycle of proper coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ for some $m \geq 2$ such that

(a) There exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) For at least one i the preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

We will proceed by induction. Notice that the case $m = 2$ follows directly from the pairwise alignment, and the case $m = 3$ follows from Lemma 4.9.2. For an inductive step, fix $m \geq 4$, and assume that there are no blocking cycles of strictly fewer than m coalitions. Let $C = \{a_1, a_2, a_3\}$.

First consider the case when $C = C_{i,i+1}$, for some $i = 1, ..., m$. Look at $C_{1,2}, C_{2,3}, C$ and conclude from Lemma 4.9.2 that either $C_{1,2} \prec_{a_1} C$, or $C_{2,3} \succ_{a_3} C$, or $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$.

- If $C = C_{i,i+1}$ and $C_{1,2} \prec_{a_1} C$ then $i \neq 1$ and the shorter cycle

  $$C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1} \prec_{a_1} C_{i,i+1}$$

  satisfies (a) and (b) because $C_{m,1} \precsim_{a_1} C_{1,2} \prec_{a_1} C = C_{i,i+1}$. This is impossible, however, by the inductive assumption.

- If $C = C_{i,i+1}$ and $C_{2,3} \succ_{a_3} C$ then $i \neq 2$ and the shorter cycle

  $$C_{i,i+1} \prec_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_i} C_{i,i+1}$$

  satisfies (a) and (b) because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$. Again, this is impossible by the inductive assumption.

- If $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$ then the cycle $C, C_{3,4} ..., C_{m,1}$ is blocking contrary to the inductive assumption.

Finally consider the case $C \neq C_{i,i+1}$ for all i. We can use the rich domain assumption to find a pairwise-aligned preference profile $\precsim_I$ such that all preferences along the blocking cycle are preserved and $C \sim_{a_1} C_{m,1}$. Abusing notation let us refer to the new profile as $\precsim_I$. Consider two subcases depending on preference of a_3 between C and $C_{2,3}$.

- If $C \prec_{a_3} C_{2,3}$, then consider the collection of $m-1$ coalitions $C, C_{3,4}, C_{4,5}, ..., C_{m,1}$. This is a blocking cycle of length $m - 1$ because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$.

- If $C \succsim_{a_3} C_{2,3}$, then consider the collection of three coalitions $C_{1,2}, C_{2,3}, C$. Since $C \sim_{a_1} C_{m,1}$, we have $C \precsim_{a_1} C_{1,2}$. Thus the collection C_1, C_2, C satisfies

  $$C \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C.$$

  By Lemma 4.9.2 all agents are then indifferent. But then $C, C_{3,4} ..., C_{m,1}$ is a blocking cycle of $m - 1$ coalitions, contrary to the inductive assumption. This completes the proof.

Proof of Theorem 4.10. Notice that if the initial profile of values was superadditive, then the proof of Lemma 4.8 and the proof of the necessity in Theorem 4.4 may be carried out while maintaining the superadditivity of the profile of values.

Proof of Theorem 4.11. By Proposition 4.5 there is a metaranking $\preccurlyeq$ on all proper coalitions. Extend this metaranking onto a relation on all coalitions by defining

$$I \preccurlyeq C \Leftrightarrow \exists \left(i \in C\right) \; I \precsim_i C$$

and verify that the extended relation $\preccurlyeq$ is still a metaranking. Now, the construction from Example 3.1 shows that whenever no agent is indifferent between two coalitions, then there is a unique core coalition structure. This lack of indifferences is generic, and thus the proof is completed.

Appendix to Section 7

Proof of Proposition 7.2. The proof of Proposition 4.6, presented in Section 4, constructs the payoff translation functions $t_{b,a} : (0, \infty) \to (0, \infty)$ for any agents a, b. Recall that for each coalition $C \ni a, b$, we have

$$t_{b,a}\left(G\left(a, C, V\right)\right) = G\left(b, C, V\right).$$

By the monotonicity of mechanism G, functions $t_{b,a}$ are strictly increasing. Since G generates Pareto optimal profiles, functions $t_{b,a}$ are continuous.

Choose an arbitrary reference agent w^* and define

$$\psi_a\left(u\right) = f\left(t_{w^*,a}\left(u\right)\right), \; a \in I$$

where $f : (0, \infty) \to R$ is continuous, decreasing, $f\left(s\right) \to +\infty$ as $s \to 0+$, and such that

$$\int_s^1 \psi_a\left(\tau\right) d\tau \to +\infty \text{ as } s \to 0+.$$

Notice that there exists a function f that satisfies these conditions. Indeed, there is a finite number k of functions $t_{w^*,a}$ which are all continuous, increasing, and have value 0 at 0. Take

$$t^{\max} = \max_a \left\{t_{w^*,a}\right\}$$

and notice that it is also continuous and increasing, and has value 0 at 0. The functions ψ_a are integrable to infinity at 0 if $f \circ t^{\max}$ is. This will be so if, for example,

$$f(t) = \frac{1}{(t^{\max})^{-1}(t)}.$$

Moreover, f is continuous and decreasing (since $s^{\max}$ is continuous and increasing), and $f(s) \to +\infty$ as $s \to 0+$ (because $s^{\max}(t) \to 0$ as $t \to 0$). Notice that ψ_a are positive and strictly decreasing and define,

$$W_a(s) = \int_1^s \psi_a(\tau)\, d\tau.$$

Now, W_a are concave and increasing, and $\lim_{s \to 0+} W_a(s) = -\infty$.

It remains to show that the solution to

$$\max_{\sum_{a\in C} \tilde{u}_a \in V} \sum_{a\in C} W_a(\tilde{u}_a) = \sum_{a\in C} \int_0^{\tilde{u}_a} \psi_a(\tau)\, d\tau$$

coincides with $G(a, C, V)$. Concavity of the problem implies that there is a solution. Since the slope at 0 for each $\int_0^{\tilde{u}_a} \psi_a(\tau)\, d\tau$ is infinite, so the solution is internal. The differentiability of the objective function implies that the internal solution is given by the first order Lagrange conditions

$$\psi_a(\tilde{u}_a) = \lambda$$

and the possibility constraint $(\tilde{u}_a)|_{a\in C} \in V$. The first order condition can be rewritten as

$$t_{w^*,a}(\tilde{u}_a) = f^{-1}(\lambda)$$

or

$$\tilde{u}_a = t_{a,w^*}\left(f^{-1}(\lambda)\right).$$

If there is no worker in C, then $C = \{f\}$ for some $f \in F$ and the claim we are proving is true. Otherwise, fix an agent $w \in C$ and notice that for agents $a \in C$

$$G(a, C, V) = t_{a,w}(G(a, C, V))$$

Lemma 4.9.2 from the appendix to section 4 shows that

$$t_{a,w^*} \circ t_{w^*,w} = t_{a,w}.$$

Hence,

$$G\left(a,C,V\right)=t_{a,w^*}\left(t_{w^*,w}\left(G\left(a,C,V\right)\right)\right)=t_{a,w^*}\left(x\right)$$

for some $x\in R$.

This equation, the analogous equation for $\tilde{u}_a$ above, the monotonicity of t_{a,w^*}, the Pareto optimality of the mechanism, and the possibility constraint $\left(\tilde{u}_a\right)|_{a\in C}\in V$ imply that

$$\tilde{u}_a=G\left(a,C,V\right).$$

This completes the proof.