Quantile Stable Mechanisms

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November 2014

Abstract

We construct *quantile stable mechanisms*, show that they are distinct in sufficiently large markets, and analyze how they can be manipulated by market participants. As a step to showing that quantile stable mechanisms are well defined, we show that median and quantile stable matchings exist when contracts are strong substitutes and satisfy the law of aggregate demand. This last result is of independent interest as experiments show that agents who match in a decentralized way tend to coordinate on the median stable matching when it exists.

1 Introduction

We consider a general matching model: there are two sides, such as firms and workers. Each agent can sign a set of contracts with agents on the other side of the market, and each agent has strict preferences over sets of contracts. Each contract specifies a firm, a worker, and the terms of matching between these two agents; they can involve

many components such as wages, benefits, etc.[1] The prominent solution concept in such matching markets is stability. If a matching is stable, then each agent is willing to keep all of her contracts and there are no additional contracts that agents would like to sign, possibly by dropping some of their current contracts.[2]

For this market, we construct *quantile stable mechanisms* and study their manipulability properties à la Pathak and Sönmez (2013). As an important step in our construction of quantile stable mechanisms we resolve the question: when do median stable matchings, and more generally, quantile stable matchings exist? This question is of independent interest because of the role of median stable matchings in decentralized matching.

Experiments show that agents who match in a decentralized way tend to coordinate on a particular stable matching, the median stable matching, when it exists, see Echenique and Yariv (2013).[3] The experimental evidence raises the question of when the median stable matching exists. Our Theorems 1 and 2 address this question, as well as a more general question of the existence of quantile stable matchings that are second best, third best, etc. for agents on the same side of the market. In particular, if there is an odd number k of stable matchings then the median stable matching is the $\frac{k+1}{2}$-th best stable matching for all agents on the chosen side.

The existence results play an auxiliary role in our construction of the quantile stable mechanisms. Two of these mechanisms are well known and often used: the extremal matching mechanisms that assign the best stable outcome for one side and the worst for the other. Extremal mechanisms have been implemented, for instance, in the National Resident Matching Program (NRMP) to match medical doctors to residency programs and in some school districts to match students to high schools

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[1]Matching markets with contracts were first studied by Roth (1984c) and Hatfield and Milgrom (2005).

[2] Stable matchings exist when contracts are *substitutes* (Roth, 1984c; Fleiner, 2003; Hatfield and Milgrom, 2005; Klaus and Walzl, 2009; Hatfield and Kominers, 2012). Contracts are substitutes when a contract that is chosen from a larger set is also chosen from a smaller set including that contract.

[3]Echenique and Yariv (2013) show that the median stable matching is selected most frequently by the subjects and that the cardinal representation of ordinal preferences also impacts which stable matching gets selected.

(Roth, 1984a; Abdulkadiroğlu and Sönmez, 2003). As far as we know, others have not been implemented despite the fact that the median stable mechanism that generates median matchings is a focal matching mechanism a market designer may want to implement: It is attractive since it may be seen as a compromise solution that treats both sides of the market in a symmetric way. Indeed, the above experimental results suggest that this is exactly how median stable matchings are perceived by experiment subjects. Furthermore, our Theorem 3 shows that no quantile stable mechanism, including deferred acceptance, is more manipulable than another when all agents are strategic. In addition, we show that quantile stable matchings are naturally ranked in terms of manipulability by both sides of the market and that the two sides rank them in an opposite way.

To present our results, let us start with the question of the existence of quantile stable matchings. Suppose that there are k stable matchings. For each agent, consider all the sets of contracts assigned to this agent in the stable matchings and rank them according to this agent's preference. We study the following questions:

1. *Existence of stable matchings*: Is the set of contracts that assigns each worker the i-th ($1 \leq i \leq k$) best stable matching outcome, say X_W^i, a matching, and is it stable?

2. *Polarity*: When does X_W^i correspond to the matching that assigns each firm the $k+1-i$-th best stable matching outcome, say X_F^{k+1-i}?

We show that two properties of agents' preferences are crucial in addressing the above questions: *strong substitutes* and *the law of aggregate demand*. Contracts are strong substitutes if a contract chosen from a set of contracts is also chosen from any worse set of contracts including that contract (Echenique and Oviedo, 2006). Contracts satisfy the law of aggregate demand if the number of chosen contracts from a larger set is weakly greater than the number of contracts chosen from a smaller set (Hatfield and Milgrom, 2005). We show that these two assumptions are satisfied in a natural model of job assignment, such as the large firms model studied in Eeckhout and Kircher (2012).

Our existence results, Theorems 1 and 2, are as follows. We show that X_W^i is a stable matching if contracts are substitutes and satisfy the law of aggregate demand for all agents, and are strong substitutes for workers (Theorem 1). On the other hand, under these conditions, X_F^{k+1-i} needs not be stable. However, we show that X_F^{k+1-i} is a stable matching and corresponds to X_W^i if contracts are also strong substitutes for firms (Theorem 2). We refer to these stable matchings as the *quantile stable matchings*. In particular, we show that if k is odd, the median stable matching outcomes for all agents can be attained simultaneously by choosing $i = (k+1)/2$ in a stable matching, which is referred to as the *median stable matching*.[4]

Two corollaries of these results are worth highlighting. First, the results allow us to show that the median and other quantile stable matchings exist and the preferences of the agents on the two sides of the market are polar when contracts are *responsive* (see Definition 5) for agents (Corollary 2).[5] The second corollary is that if contracts are strong substitutes then the median stable matching and other quantile stable matchings exist, and are polar, in many-to-one matching markets with wages when agents have quasilinear utility and contracts are written over salaries (Kelso and Crawford, 1982). These corollaries are new even for the median stable matching. In both corollaries, we do not need to impose the law of aggregate demand explicitly since in the former case responsiveness trivially implies the law of aggregate demand, and, in the latter case, quasilinear utility with substitutability deliver the law of aggregate demand (Hatfield and Milgrom, 2005). In the Appendix, through examples, we show that quantile stable matchings need not exist if we weaken strong

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[4]Methodologically, we build on the lattice structure of stable matchings (Fleiner, 2003). Fleiner (2003) shows that under substitutes and the law of aggregate demand, the set of stable matchings forms a lattice. In particular, the chosen set of contracts by firms (or workers) out of the union of two stable matchings is, itself, a stable matching. This result would imply the quantile stable matching structure if we knew that the lattice operator coincides with the supremum for firms (or workers). While it is always so in one-to-one matching, this latter property may fail in many-to-one and many-to-many matching markets. It is one of our contributions to recognize that the lattice operator coincides with the supremum for firms (or workers) when strong substitutes and the law of aggregate demand are imposed.

[5]We derive this corollary even though responsiveness is neither implied by nor implies strong substitutability; see the discussion in Echenique and Oviedo (2006, Section 6.3). Of course, the existence of the median stable matching requires k to be odd.

substitutes or the law of aggregate demand.

Having established the existence of quantile stable matchings, we analyze the resulting quantile stable mechanisms. For any $q \in (0, 1]$ the q-quantile stable mechanism maps agents' preference profiles into the $\lceil kq \rceil$ best stable matching for either firms or workers, where k is the number of stable matchings.[6] Examples of such mechanisms include the mechanism that always selects the worker-optimal stable matching, as well as a mechanism that among k stable matchings always selects the matching that is the $\lceil \frac{k}{2} \rceil$-best for workers (and hence $\lfloor \frac{k}{2} \rfloor$-best for firms).

Building on Pathak and Sönmez (2013) and Chen et al. (2014a), we study manipulability of the quantile stable mechanisms. We say that a mechanism ψ is *as manipulable as* mechanism ϕ for an agent if whenever the agent can gain from misreporting in ϕ and achieve a certain outcome, she can also gain and achieve this outcome by manipulating ψ. We say that mechanism ψ is *more manipulable* than mechanism ϕ for an agent if it is as manipulable and in addition there exists an instance of the market in which she can manipulate ψ but not ϕ. Our polarity result discussed above implies that quantile stable mechanisms can be naturally ranked for each side of the market in terms of how manipulable they are. In particular, as we choose a higher quantile for one side of the market, we make the mechanism less manipulable for that side, but more manipulable for the other side of the market (note that the mechanism is then choosing a lower quantile for the other side of the market) (Theorem 3). Thus, no quantile stable matching is better than another in terms of manipulability when all agents in the market are strategic.

To the best of our knowledge ours is the first paper to study quantile stable mechanisms other than deferred acceptance, and, in particular, the first to study their incentive properties. However, the existence of quantile stable matchings—which plays an auxiliary role in our results on quantile stable mechanisms—has been previously established in some special cases of our setting: the college admissions model with responsive preferences (Klaus and Klijn, 2006; Sethuraman et al., 2006), and one-to-one matching (Teo and Sethuraman, 1998; Fleiner, 2002; Schwarz and

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[6] $\lceil x \rceil$ is the smallest integer that is weakly larger than x.

Yenmez, 2011).[7] All of these results are implied by our more general treatment; moreover, the two corollaries highlighted earlier are new. In particular, Schwarz and Yenmez (2011) discuss the substantial challenges involved in trying to address the existence of quantile stable matchings for many-to-one matching markets with wages; they leave the question open. Our Theorems 1 and 2 go beyond this prior literature also by identifying the forces behind the quantile structure and polarity results.[8]

2   Model

There are two sets of agents: the set of firms F, and the set of workers W. The set of all agents is denoted by $A \equiv F \cup W$. Each contract x is bilateral and specifies the relationship between a firm-worker pair. The firm and worker associated with contract x are represented by $x_F \in F$ and $x_W \in W$, respectively. The set of all contracts is finite and denoted by X. For a set of contracts $X' \subseteq X$, $X'_a \equiv \{x | x \in X', a \in \{x_F, x_W\}\}$ denotes the set of contracts that agent a is associated with. A set of contracts X' is **feasible** if for every firm-worker pair f, w, $|X'_f \cap X'_w| \leq 1$, i.e., each firm-worker pair can sign at most one joint contract. A **matching** is a feasible set of contracts.

Each agent a is endowed with a strict preference relation $\succ_a$ over sets of contracts that involve agent a, i.e., over $2^{X_a} \equiv \{X' | X' \subseteq X_a\}$. Similarly, agent a's weak preference relation is denoted by $\succeq_a$, so for all $Y, Y' \subseteq X_a$, $Y \succeq_a Y'$ if and only if $Y \succ_a Y'$ or $Y = Y'$. Given $\succeq_a$, let $C_a(X' | \succeq_a)$ denote agent a's most preferred subset of contracts involving agent a from X'. More formally, $C_a(X' | \succeq_a) \subseteq X'_a$ and for all $Y_a \subseteq X'_a$, $C_a(X' | \succeq_a) \succeq_a Y_a$. To ease notation, we suppress the dependence on $\succeq_a$

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[7]There are two exceptions. First, Klaus and Klijn (2010) study the existence of quantile stable matchings in the roommates problem. Second, Schwarz and Yenmez (2011) study the case when there is a continuum of potential wages in addition to the finite case.

[8]Our results also show that even when quantile stable matchings do not exist for all agents, they may still exist for one side of the market if contracts are strong substitutes (or preferences are responsive) for that side. In particular, when workers have unit demand, i.e., for all $w \in W$ and $Y \subseteq X$ $|C_w(Y)| \leq 1$, contracts are strong substitutes for workers automatically and, therefore, worker quantile stable matchings exist if contracts are substitutes and satisfy the law of aggregate demand for firms.

throughout the paper (when this does not lead to ambiguity) and denote the set of contracts chosen from $X' \subseteq X$ by $C_a(X')$. Similarly, for any set of contracts X', let $C_W(X') \equiv \cup_w C_w(X')$ and $C_F(X') \equiv \cup_f C_f(X')$ be the chosen sets of contracts for the set of workers and firms, respectively. We say that a contract x is **acceptable** to agent a if there exists a set of contracts $X_a \ni x$ such that $X_a \succ_a \emptyset$; otherwise we say that contract x is unacceptable to a.

Given agents and their preferences, we would like to find a matching that no set of agents would like to deviate from. This is formalized in the following definition of stability.

Definition 1. *Given a preference profile $\succ$, a matching Y is **stable** if*

1. *for all a, $C_a(Y) = Y_a$ (**individual rationality**) and*

2. *there does not exist a nonempty set of contracts $Z \nsubseteq Y$ such that for all a, $Z_a \subseteq C_a(Y \cup Z)$ (**no blocking**).*

Stability for a matching entails two things: Individual rationality requires that each agent is better off by holding all of the contracts assigned rather than rejecting some of them. On the other hand, no blocking states that there is no subset of contracts Z such that every agent a would choose Z_a if Z is available to them. This is the standard definition of stability for many-to-many matching with contracts: see Hatfield and Kominers (2012).

We make the following assumptions on agents' preferences in our analysis.

Definition 2. *Contracts are **substitutes** in preferences of agent a if for any sets of contracts $Y, Y' \subseteq X$ such that $Y \subseteq Y'$ and a contract x*

$$x \in C_a(Y' \cup \{x\}) \Rightarrow x \in C_a(Y \cup \{x\}).$$

Contracts are substitutes if a contract that is chosen from a larger set is still chosen from a smaller set including the contract. Substitutability is standard in

matching: see Kelso and Crawford (1982); Roth (1984c). It guarantees the existence of a stable matching in our setup (Fleiner, 2003).[9]

Definition 3. *Contracts are* **strong substitutes** *in preferences of agent* a *if for any sets of contracts* $Y, Y' \subseteq X$ *such that* $C_a(Y') \succeq_a C_a(Y)$,

$$x \in C_a(Y' \cup \{x\}) \Rightarrow x \in C_a(Y \cup \{x\}).$$

Strong substitutability implies substitutability. Roughly, it states that if a contract is added to the two sets and chosen from the better set, then it must also be chosen from the worse set. Echenique and Oviedo (2006) introduced strong substitutability for matching markets without contracts.[10]

Definition 4. *Contracts satisfy the* **law of aggregate demand** *in preferences of agent* a *if for all* $Y, Y' \subseteq X$ *such that* $Y \subseteq Y'$

$$|C_a(Y)| \leq |C_a(Y')|.$$

The law of aggregate demand requires that the number of contracts chosen from a set is bigger than the number of contracts chosen from a subset of this set. The law of aggregate demand was introduced in Alkan (2002), Alkan and Gale (2003), and Fleiner (2003) and its implications were thoroughly analyzed by Hatfield and Milgrom (2005) (see also Kojima (2007)).

Although the class of environments in which contracts are strong substitutes and satisfy the law of aggregate demand is limited, it encompasses some interesting examples. For instance, Hatfield and Milgrom (2005) show that if contracts specify monetary payments and agents' preferences are quasi-linear, then the law of aggregate demand is satisfied. Hatfield and Milgrom (2005) also provide the endowed

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[9]Aygün and Sönmez (2013) show that substitutability alone does not guarantee the existence of stable matchings when choice rules are taken as primitives and prove that an axiom called *irrelevance of rejected contracts* is needed for the existence. This axiom is satisfied in our setup since choice rules are constructed using strict preferences over sets of contracts.

[10]As in Echenique and Oviedo (2006), our results would remain valid if we had the strong substitutes condition require only that for agent a if for any sets of contracts $Y, Y' \subseteq X$ such that $C_a(Y') = Y'_a$, $C_a(Y) = Y_a$, and $C_a(Y') \succeq_a C_a(Y)$, we had $x \in C_a(Y' \cup \{x\})$ then $x \in C_a(Y \cup \{x\})$.

assignment model in which contracts are substitutes in addition to satisfying the law of aggregate demand. The example below shows that in a natural special case of the endowed assignment model contracts are not only substitutes but also strong substitutes.

Example 1. Consider a many-to-one matching market between firms and workers. Each firm has a technology and a set of jobs to fill. A worker can fill any of a firm's jobs but no firm can use one worker for two jobs, nor assign two or more workers to the same job. A contract specifies the firm, the job, the worker, and the wage transfer. Each worker-job pair generates an output; this output does not depend on what other workers the firm employs or how it assigns them to jobs. The firm's payoff is the sum of outputs from its different jobs, net of wage transfers. If all available contracts for a particular job at a firm yield negative net output, then the firm leaves that job unfilled. Workers choose among contract based on the wages offered; if no non-negative wage contract is available, then the worker accepts none.

So far we have described the endowment assignment model of Hatfield and Milgrom (2005). In their model contracts are substitutes and satisfy the law of aggregate demands, but the strong substitutes condition may fail. To guarantee that this last condition is satisfied, we assume that each worker is characterized by a one-dimensional ability parameter. If the worker's output in a job depends only on the job and the worker's ability (but not on his identity otherwise), and if output is increasing in worker's ability, then contracts are strong substitutes.[11]

3 Results

In this section, we introduce quantile stable mechanisms and compare them in terms of manipulability. First, we construct quantile stable matchings and show that they exist if contracts are strong substitutes and satisfy the law of aggregate demand.

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[11]We can generalize this example to a many-to-many matching market. Eeckhout and Kircher (2012) study a continuous version of this setting. In particular, taking the limit of our model in the case of quasi-linear preferences with monetary increments, like in Schwarz and Yenmez (2011), one can use our results to show that the set of competitive equilibria in Eeckhout and Kircher (2012) has a quantile structure.

3.1 Quantile Stable Matchings

Suppose that $\{X^1, \ldots, X^k\}$ is the set of stable matchings. For each agent a, consider the set of contracts that agent a signs in these matchings: $\{X_a^1, \ldots, X_a^k\}$. Reorder these sets of contracts according to $\succeq_a$ such that $X_a^{(1)} \succeq_a \ldots \succeq_a X_a^{(k)}$. Let $X_F^i \equiv \underset{f \in F}{\cup} X_f^{(i)}$ and $X_W^i \equiv \underset{w \in W}{\cup} X_w^{(i)}$ for $1 \leq i \leq k$. In words, X_F^i assigns each firm the i-th best outcome among all stable matching outcomes and X_W^i assigns each worker the i-th best outcome among all stable matching outcomes. We analyze when these sets of contracts are stable.

Theorem 1. *Suppose that contracts are substitutes and satisfy the law of aggregate demand for all agents. Suppose also that contracts are strong substitutes for workers. Then for all i, X_W^i is a stable matching. Moreover, for any firm f, $X_W^i \succeq_f X_W^j$ if $i \geq j$.*

Under the conditions of Theorem 1, we call X_W^i the i-**th quantile stable matching for workers**. In addition, X_W^1 is called the **worker-optimal stable matching** and X_W^k is the **worker-pessimal stable matching**. In the symmetric case, when contracts are strong substitutes for firms, we define the quantile, firm-optimal, and firm-pessimal stable matchings for firms. The worker-optimal and firm-optimal stable matchings still exist when contracts are just substitutes and satisfy the law of aggregate demand for all agents. This follows from the lattice structure shown in Alkan (2002) and Fleiner (2003). However, the other quantile stable matchings need not exist: see Example 2 in the appendix.

A direct consequence of this result is that when workers have unit demand quantile stable matchings exist. This answers the open question posed in Schwarz and Yenmez (2011) about the existence of quantile stable matchings in many-to-one matching markets when contracts are written over wages when there is a finite set of contracts.

Corollary 1. *Suppose that workers have unit demand, i.e., for all $w \in W$ and $Y \subseteq X$ $|C_w(Y)| \leq 1$. Suppose also that contracts are substitutes and satisfy the law*

of aggregate demand for firms. Then for all i, X_W^i is a stable matching. Moreover, for any firm f, $X_W^i \succeq_f X_W^j$ if $i \geq j$.

Next we show that quantile stable matchings for firms and workers are exactly the same with the polarization of interests property when contracts are also strong substitutes for all firms.

Theorem 2. *Suppose that contracts are strong substitutes and satisfy the law of aggregate demand for all agents. Then for all i, X_W^i and X_F^i are stable matchings; moreover, $X_W^i = X_F^{k+1-i}$.*

As a corollary we can also obtain a result for the setting with responsive preferences.

Definition 5. *Contracts are **responsive** for agent a if there exist a quota q_a and a strict relation $>_a$ on contracts X_a and the empty contract $\emptyset$ such that (1) agent a prefers the empty contract to any set of contracts $|Y| > q_a$, and (2) agent a weakly prefers a set of contracts $Y \equiv \{y_1, \ldots, y_{|Y|}\} \subseteq X_a$ to another set of contracts $Z \equiv \{z_1, \ldots, z_{|Z|}\} \subseteq X_a$ whenever*

- $q_a \geq |Y| \geq |Z|$ *and* $y_i >_a z_i$ *for every* $1 \leq i \leq |Z|$ *and* $y_i >_a \emptyset$ *for every* $|Z| + 1 \leq i \leq |Y|$, *or*

- $|Y| \leq |Z|$ *and* $y_i >_a z_i$ *for every* $1 \leq i \leq |Y|$ *and* $\emptyset >_a z_i$ *for every* $|Y| + 1 \leq i \leq |Z|$.

If contracts are responsive for agent a, then there is an order over individual contracts and from each set of contracts agent a chooses the best contracts without exceeding its quota. A contract x is **acceptable** for agent a if $x >_a \emptyset$. Note that substitutes and the law of aggregate demand are implied by responsiveness.

Corollary 2. *Suppose that contracts are responsive for all agents. Then for all i, X_W^i and X_F^i are stable matchings; moreover, $X_W^i = X_F^{k+1-i}$.*

To reduce this corollary to Theorem 2, one can follow step by step the argument in Roth and Sotomayor (1990) in which they prove that the classical lattice structure

of the marriage problem carries over to the college admissions problem when colleges have responsive preferences.[12]

If contracts are strong substitutes and satisfy the law of aggregate demand, then the quantile stable matchings for both firms and workers exist. Moreover, these matchings are aligned in the following way: the worker-optimal stable matching is the firm-pessimal stable matching, the (2)-nd quantile stable matching for workers is the $(k - 1)$-th quantile stable matching for firms, etc. In particular, when k is odd, there exists a stable matching that assigns all agents their median stable matching outcomes since $X_F^{(k+1)/2} = X_W^{(k+1)/2}$.

We provide examples in the Appendix which demonstrate that the quantile stable matchings need not exist if we weaken strong substitutability to substitutability or get rid of the law of aggregate demand.

3.2 Quantile Stable Mechanisms

Finally, let us define the quantile stable mechanisms and examine their incentive properties. Fix one side of the matching market, say firms. For each $q \in (0, 1]$, the **q-quantile stable mechanism** φ^q is the mapping from agents' preference profiles to matchings such that for every preference profile $\succ$, the mechanism $\varphi^q(\succ)$ selects the $\lceil kq \rceil$-th quantile stable matching for firms where k is the number of stable matchings under $\succ$. Here, $\lceil x \rceil$ denotes the lowest integer equal to or larger than x.[13]

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[12]We first create an auxiliary one-to-one matching problem with contracts. In this auxiliary problem, the set of agents consists of q_a numbered copies of each agent a from the original problem; for each pair of replicas of agents f and w from two sides of the market, the set of contracts they can sign is isomorphic to the set of contracts f and w can sign in the original problem. We construct agents' preferences in the auxiliary problem as follows. Take an agent $\hat{a}$, who is a replica of agent a in the original problem. Take any two contracts $\hat{x}$ and $\hat{y}$ created from contracts $x, y \in X_a$. If $x \succ_a y$ in the original problem, then $\hat{x} \succ_{\hat{a}} \hat{y}$ in the auxiliary problem. If x is identical to y in the original problem, then the contracts $\hat{x}$ and $\hat{y}$ are with two different numbered replicas of the same original-problem-agent, and agent $\hat{a}$ ranks the two contracts according to the numbers of the replicas. For the auxiliary one-to-one matching problem with contracts, Theorem 2 implies that there is a lattice structure on stable matchings. Following the same steps as Roth and Sotomayor (1990) we then conclude that this lattice induces a lattice on stable matchings in the original problem.

[13]All our results remain valid for mechanisms that always select the $\lfloor kq \rfloor$-th quantile stable matching, where $\lfloor x \rfloor$ is the highest integer smaller than or equal than x.

Theorem 2 shows that the quantile stable mechanisms are well defined when we restrict attention to the domain of preferences that satisfy strong substitutes and the law of aggregate demand. Corollary 2 shows that they are also well defined on the domain of responsive preferences. In what follows we assume that the quantile stable mechanisms we study are defined on one of these two preference domains.

Importantly, each of these two preference domains is closed in the sense of Chen et al. (2014a): A preference profile domain P is **closed** if for all $\succ \in P$ and for all matchings Y that are stable with respect to $\succ$, if the preference relation $\succ'_a$ ranks sets of contracts in the same way as $\succ_a$ except that only contracts in Y_a are acceptable to agent a, then $(\succ'_a, \succ_{-a}) \in P$. This allows us to rely on their results.

We define manipulability as in Chen et al. (2014a); these definitions are based on Pathak and Sönmez (2013).[14]

Definition 6. *Mechanism ψ is **as manipulable as** mechanism ϕ for agent a if for any preference profile $\succ \in P$, the following holds: if there exist agent a and preference relation $\succ'_a \in P_a$ such that $\phi(\succ'_a, \succ_{-a})(a) \succ_a \phi(\succ)(a)$, then there exists a preference relation $\succ''_a \in P_a$ such that $\psi(\succ''_a, \succ_{-a})(a) = \phi(\succ'_a, \succ_{-a})(a)$ and $\psi(\succ''_a, \succ_{-a})(a) \succ_a \psi(\succ)(a)$.*

Definition 7. *Mechanism ψ is **more manipulable than** mechanism ϕ for agent a if ψ is as manipulable as ϕ for agent a and there is a profile of preference relations $\succ \in P$ at which agent a can manipulate ψ but not ϕ; that is, for all preference relations $\succ'_a \in P_a$ we have $\phi(\succ)(a) \succeq_a \phi(\succ'_a, \succ_{-a})(a)$, and there exists a preference profile $\succ''_a \in P_a$ such that $\psi(\succ''_a, \succ_{-a})(a) \succ_a \psi(\succ)(a)$.*

Our polarity result and Theorem 2 in Chen et al. (2014a) imply the following.

Theorem 3. *Let $q, q' \in (0, 1]$ be such that $q > q'$. Then either*

- *$\varphi^q = \varphi^{q'}$, or*

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[14]The definition of more manipulability in Chen et al. (2014a) is slightly more demanding then the analogous definition in Pathak and Sönmez (2013) (see Chen et al. (2014a) for a discussion). We formulate our Theorem 3 for the more demanding definition, but, a fortiori, it remains valid for the less demanding one.

- φ^q is more manipulable than $\varphi^{q'}$ for all firms and $\varphi^{q'}$ is more manipulable than φ^q for all workers.

Finally, we show the following complementary result.

Theorem 4. *For any $q, q' \in (0, 1]$ such that $q \neq q'$, there exists a matching market such that φ^q is different than $\varphi^{q'}$.*

Proof. Assume that $q > q'$ without loss of generality. Let k be such that $k(q-q') > 1$. Consider the following market.

All agents have unit demand and every firm-worker pair uniquely defines a contract. Let firm f_i rank workers as follows $w_i \succ_{f_i} w_{i+1} \succ_{f_i} \ldots \succ_{f_i} w_{i+k-1}$ (subscripts are added modulo k) and let worker w_{i+k-1} rank firms as follows $f_i \succ_{w_{i+k-1}} f_{i-1} \succ_{w_{i+k-1}} \ldots \succ_{w_{i+k-1}} f_{i-k+1}$. Under this preference profile there are k distinct stable matchings and all quantile stable matchings are different.

In this market, the q-quantile stable mechanism is different from the q'-quantile stable mechanism since $k(q - q') > 1$. $\qquad\square$

Appendix: Omitted Proofs and Examples

First, we provide two lemmas that are used in the proofs of Theorems 1 and 2.

There exists at least one stable matching once we impose that contracts are substitutes (Fleiner, 2003; Klaus and Walzl, 2009; Hatfield and Kominers, 2012). Here, we study the structure of the set of stable matchings if, in addition to substitutability, contracts satisfy the law of aggregate demand. In particular, we are interested in when this set is a lattice with respect to the following operators.

Let Y and Y' be two sets of contracts. Define the following sets of contracts:

$$Y \vee_F Y' = \bigcup_f \max_{\succeq_f} \{Y_f, Y'_f\},$$

and

$$Y \wedge_F Y' = \bigcup_f \min_{\succeq_f} \{Y_f, Y'_f\}.$$

Operator $\vee_F$ chooses the most preferred set of contracts for each firm. On the other hand, $\wedge_F$ chooses the least preferred set of contracts for each firm. Analogously, we define $Y \vee_W Y'$ and $Y \wedge_W Y'$:

$$Y \vee_W Y' = \bigcup_w \max_{\succeq_w}\{Y_w, Y'_w\},$$

and

$$Y \wedge_W Y' = \bigcup_w \min_{\succeq_w}\{Y_w, Y'_w\}.$$

By definition, all of these operators define a set of contracts but in general they do not have to be stable matchings. We study this question in the following lemma.

Lemma 1. *Suppose that contracts are substitutes and satisfy the law of aggregate demand for all agents. Suppose also that contracts are strong substitutes for workers. Then, for any two stable matchings Y and Y', $Y \vee_W Y'$ and $Y \wedge_W Y'$ are stable matchings. Moreover, for each firm f, $(Y \wedge_W Y') \succeq_f Y, Y'$ and $Y, Y' \succeq_f (Y \vee_W Y')$.*

Proof. [Proof of Lemma 1] Let $x \in Y \vee_W Y'$, so there exists w such that $x \in \max_{\succeq_w}\{Y_w, Y'_w\}$. We want to show that $x \in C_w(Y \cup Y')$. If $x \in Y_w \cap Y'_w$ then the claim follows from the following observation that is implied by Corollary 26 and Equation 38 in Fleiner (2003).

Observation 1. Suppose that contracts are substitutes and satisfy the law of aggregate demand for all agents. If Y and Y' are stable matchings, then $C_W(Y \cup Y')$ and $C_F(Y \cup Y')$ are stable matchings such that

$$Y \cup Y' = C_F(Y \cup Y') \cup C_W(Y \cup Y') \quad \text{and} \quad Y \cap Y' = C_F(Y \cup Y') \cap C_W(Y \cup Y').$$

Let us thus suppose $x \in Y_w$ and $x \notin Y'_w$. This implies that $Y_w \succeq_w Y'_w$.

Since Y is stable, we have $x \in Y_w = C_w(Y_w) = C_w(Y_w \cup \{x\})$. Thus $Y_w \succeq_w Y'_w$ and strong substitutes imply $x \in C_w(Y'_w \cup \{x\})$. Let f be x_F. If $x \in C_f(Y'_f \cup \{x\})$, then $\{x\}$ would block Y', which contradicts the stability of Y'. Therefore, $x \notin C_f(Y'_f \cup \{x\})$. Substitutability then implies that $x \notin C_f(Y'_f \cup Y_f)$ and $x \notin C_F(Y' \cup Y)$. The

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observation highlighted above then implies $x \in C_W(Y \cup Y')$. Hence $Y \vee_W Y' \subseteq C_W(Y \cup Y')$.

Since Y, Y', and $C_W(Y' \cup Y)$ are stable matchings, the rural hospital theorem implies $|Y_{w'}| = |Y'_{w'}| = |C_{w'}(Y \cup Y')|$ for every worker w'.[15] By construction, for every worker w', $|(Y \vee_W Y')_{w'}| = |Y_{w'}| = |Y'_{w'}|$, so $|Y \vee_W Y'| = |Y| = |Y'| = |C_W(Y \cup Y')|$. The inclusion proven above allows us to conclude that $Y \vee_W Y' = C_W(Y \cup Y')$. Therefore, $Y \vee_W Y'$ is a stable matching.

Denote by χ the indicator function on sets of contracts. By the observation highlighted above, $\chi(Y) + \chi(Y') = \chi(C_F(Y \cup Y')) + \chi(C_W(Y \cup Y'))$ and, by definition, $\chi(Y \vee_W Y') + \chi(Y \wedge_W Y') = \chi(Y) + \chi(Y')$. We thus get $\chi(Y \vee_W Y') + \chi(Y \wedge_W Y') = \chi(C_F(Y \cup Y')) + \chi(C_W(Y \cup Y'))$. Above we have shown that $Y \vee_W Y' = C_W(Y \cup Y')$, so $\chi(Y \vee_W Y') = \chi(C_W(Y \cup Y'))$. Therefore, $\chi(Y \wedge_W Y') = \chi(C_F(Y \cup Y'))$ and thus $Y \wedge_W Y' = C_F(Y \cup Y')$, so $Y \wedge_W Y'$ is a stable matching.

The last claim of the lemma now follows similarly to the polarization of interest property established by Echenique and Oviedo (2006) (Theorem 9.8); while they derive the polarization of interests for many-to-many matching markets without contracts, an analogous argument works in the setting with contracts we study. $\qquad\square$

Next we impose that contracts are strong substitutes for all agents to show that $\vee_F = \wedge_W$ and $\vee_W = \wedge_F$.

Lemma 2. *Suppose that contracts are strong substitutes and satisfy the law of aggregate demand for all agents. Then for any two stable matchings Y and Y', $Y \vee_F Y' = Y \wedge_W Y'$, $Y \wedge_F Y' = Y \vee_W Y'$. Moreover, $Y \vee_F Y'$ and $Y \wedge_F Y'$ are stable matchings.*

Proof. This result follows directly from Lemma 1 above. In the proof of Lemma 1, we have shown that $Y \vee_W Y'$ and $Y \wedge_W Y'$ are stable matchings, $Y \vee_W Y' = C_W(Y \cup Y')$, and $Y \wedge_W Y' = C_F(Y \cup Y')$, relying on the assumption that contracts are strong

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[15] The rural hospitals theorem says that any agent signs the same number of contracts in any stable matching when contracts are substitutes and satisfy the law of aggregate demand; in the setting of our lemma it was proved by Alkan (2002); Fleiner (2003); Hatfield and Kominers (2012), see also Roth (1984b).

substitutes for workers. Symmetrically, since contracts are strong substitutes for firms, $Y \vee_F Y'$ and $Y \wedge_F Y'$ are stable matchings, where $Y \vee_F Y' = C_F(Y \cup Y')$, and $Y \wedge_F Y' = C_W(Y \cup Y')$. Therefore, $Y \vee_W Y' = Y \wedge_F Y'$ and $Y \wedge_W Y' = Y \vee_F Y'$. $\square$

Proof. [Proofs of Theorems 1 and 2] First we prove that if $Y \vee_W Y'$ and $Y \wedge_W Y'$ are stable matchings for any two stable matchings Y and Y', then X_W^i is also a stable matching. This proves the first part of Theorem 1 and the first part of Theorem 2. The second part of Theorem 1 follows directly from Lemma 1. Next we prove the second part of Theorem 2: $X_W^i = X_F^{k+1-i}$.

Consider all combinations of i sets of stable matchings $Y^1, \ldots, Y^i$ where $1 \le i \le k$. For each combination consider $Y^1 \wedge_W \ldots \wedge_W Y^i$, which assigns each worker the least preferred set of contracts. There are $l \equiv \binom{k}{i}$ of these stable matchings; denote them by $Z^1, \ldots, Z^l$. Finally, let $\xi^i \equiv Z^1 \vee_W \ldots \vee_W Z^l$ be the set of contracts that assigns each worker the best set of contracts among available ones. By Lemma 1, ξ^i is a stable matching. We claim that $\xi^i = X_W^i$.

For each Z^j and w, $Z_w^j \preceq_w X_w^{(i)}$ by construction of Z^j. Similarly, there exists j^* such that $Z_w^{j^*} = X_w^{(i)}$. Therefore, for all w, $\xi_w^i = X_w^{(i)}$, which implies $\xi^i = X_W^i$.

To finish the proof we show $X_W^i = X_F^{k+1-i}$ under the conditions of Theorem 2. By Lemma 2, $Y \vee_F Y' = Y \wedge_W Y'$, $Y \wedge_F Y' = Y \vee_W Y'$. Therefore, $Y^1 \vee_W \ldots \vee_W Y^i = Y^1 \wedge_F \ldots \wedge_F Y^i$. Therefore, for each Z^j and f, $Z_f^j \succeq_f X_f^{(k+1-i)}$ by construction of Z^j. Similarly, there exists j^* such that $Z^{j^*} = X_f^{(k+1-i)}$. Since $\xi^i \equiv Z^1 \wedge_W \ldots \wedge_W Z^l = Z^1 \vee_W \ldots \vee_W Z^l$ by Lemma 2 $\xi^i = X_F^{k+1-i}$, which implies $X_W^i = X_F^{k+1-i}$. $\square$

In the following example, we show that strong substitutes cannot be replaced with substitutes in Theorems 1 and 2.

Example 2. There are two firms f_1, f_2; and four workers w_1, w_2, w_3, and w_4. Workers have unit demand. There is only one contract that each firm-worker pair can sign. To ease notation, each contract is denoted by the pair of agents associated with this contract. Let $x_1 = \{f_1, w_1\}$, $x_2 = \{f_1, w_2\}$, $x_3 = \{f_1, w_3\}$, $x_4 = \{f_1, w_4\}$, $x_5 = \{f_2, w_1\}$, $x_6 = \{f_2, w_2\}$, $x_7 = \{f_2, w_3\}$, and $x_8 = \{f_2, w_4\}$. Preferences are as

follows:[16]

$$\succeq_{f_1} \quad : \quad x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_1, x_3, x_2, x_4;$$

$$\succeq_{f_2} \quad : \quad x_6 x_8, x_5 x_8, x_6 x_7, x_5 x_7, x_6, x_8, x_5, x_7;$$

$$\succeq_{w_1} \quad : \quad x_5, x_1;$$

$$\succeq_{w_2} \quad : \quad x_2, x_6;$$

$$\succeq_{w_3} \quad : \quad x_7, x_3; \text{ and}$$

$$\succeq_{w_4} \quad : \quad x_4, x_8.$$

Note that contracts are substitutes and satisfy the law of aggregate demand for all agents. Moreover, they are also strong substitutes for workers and f_2. However, contracts are not strong substitutes for f_1 because even though $x_1 x_4 \succeq_{f_1} x_2 x_3$, and $x_4 \in C_{f_1}(\{x_1, x_4\} \cup \{x_4\})$ we have $x_4 \notin C_{f_1}(\{x_2, x_3\} \cup \{x_4\}) = \{x_2, x_3\}$.

There are four stable matchings: $\mu_1 \equiv \{x_1, x_3, x_6, x_8\}$, $\mu_2 \equiv \{x_1, x_4, x_6, x_7\}$, $\mu_3 \equiv \{x_2, x_3, x_5, x_8\}$, and $\mu_4 \equiv \{x_2, x_4, x_5, x_7\}$. When we use the quantile construction above we get: $X_F^1 = \{x_1, x_3, x_6, x_8\} = \mu_1$, $X_F^2 = \{x_1, x_4, x_5, x_8\}$, $X_F^3 = \{x_2, x_3, x_6, x_7\}$, $X_F^4 = \{x_2, x_4, x_5, x_7\} = \mu_4$; and $X_W^1 = \{x_2, x_4, x_5, x_7\} = \mu_1$, $X_W^2 = \{x_2, x_4, x_5, x_7\} = \mu_1$, $X_W^3 = \{x_1, x_3, x_6, x_8\} = \mu_4$, $X_W^4 = \{x_1, x_3, x_6, x_8\} = \mu_4$. Here X_F^2 and X_F^3 are not even matchings, let alone stable.

Finally, we show that the law of aggregate demand is also necessary in Theorems 1 and 2, that is, the law of aggregate demand is necessary for the existence of quantile matchings. In the following example contracts are strong substitutes for all agents but the law of aggregate demand fails. There are four stable matchings including the firm-optimal and the worker-optimal stable matchings, but other quantile stable matchings do not exist.[17]

Example 3. There are five firms f_1, f_2, f_3, f_4, f_5; and five workers w_1, w_2, w_3, w_4, and

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[16]A set of contracts $\{x_i, \ldots, x_j\}$ is denoted by $x_i \ldots x_j$ to ease notation. If a set of contracts is omitted from the preference list of an agent, the agent prefers the null contract to that set of contracts.

[17]This example develops Example 5 of Alkan and Gale (2003) who use it to show that the law of aggregate demand is necessary to get the lattice structure of stable matchings.

w_5. Workers have unit demand. For each firm-worker pair, there is only one contract that they can sign. To ease notation, each contract is denoted by the pair of agents associated with this contract. Let $x_1 = \{f_1, w_1\}$, $x_2 = \{f_1, w_3\}$, $x_3 = \{f_1, w_5\}$, $x_4 = \{f_2, w_2\}$, $x_5 = \{f_2, w_4\}$, $x_6 = \{f_2, w_5\}$, $x_7 = \{f_3, w_3\}$, $x_8 = \{f_3, w_1\}$, $x_9 = \{f_4, w_4\}$, $x_{10} = \{f_4, w_2\}$, and $x_{11} = \{f_5, w_5\}$. Preferences are as follows:

$$\succeq_{f_1} \; : \; x_1, x_2 x_3, x_2, x_3;$$

$$\succeq_{f_2} \; : \; x_4, x_5 x_6, x_5, x_6;$$

$$\succeq_{f_3} \; : \; x_7, x_8;$$

$$\succeq_{f_4} \; : \; x_9, x_{10};$$

$$\succeq_{f_5} \; : \; x_{11};$$

$$\succeq_{w_1} \; : \; x_8, x_1;$$

$$\succeq_{w_2} \; : \; x_{10}, x_4;$$

$$\succeq_{w_3} \; : \; x_2, x_7$$

$$\succeq_{w_4} \; : \; x_5, x_9; \text{ and}$$

$$\succeq_{w_5} \; : \; x_3, x_6, x_{11}.$$

Since workers f_3, f_4, and f_5 have unit demand, contracts are strong substitutes and satisfy the law of aggregate demand for these agents. However, even though contracts are strong substitutes for f_1 and f_2, they do not satisfy the law of aggregate demand for these two firms. This is rather straightforward. For example, for f_1, $C_{f_1}(\{x_1, x_2, x_3\}) = x_1$ and $C_{f_1}(\{x_2, x_3\}) = \{x_2, x_3\}$, which imply $|C_{f_1}(\{x_1, x_2, x_3\})| < |C_{f_1}(\{x_2, x_3\})|$.

There are four stable matchings: $\mu_1 \equiv \{x_1, x_4, x_7, x_9, x_{11}\}$, $\mu_2 \equiv \{x_1, x_5, x_6, x_7, x_{10}\}$, $\mu_3 \equiv \{x_2, x_3, x_4, x_8, x_9\}$, and $\mu_4 \equiv \{x_2, x_3, x_5, x_8, x_{10}\}$. When we use the quantile construction above we get: $X_F^1 = \{x_1, x_4, x_7, x_9, x_{11}\} = \mu_1$, $X_F^2 = \{x_1, x_4, x_7, x_9\}$, $X_F^3 = \{x_2, x_3, x_5, x_6, x_8, x_{10}\}$, $X_F^4 = \{x_2, x_3, x_5, x_8, x_{10}\} = \mu_4$; and $X_W^1 = \{x_2, x_3, x_5, x_8, x_{10}\} = \mu_4$, $X_W^2 = \{x_2, x_3, x_5, x_8, x_{10}\} = \mu_4$, $X_W^3 = \{x_1, x_4, x_6, x_7, x_9\}$, $X_W^4 = \{x_1, x_4, x_7, x_9, x_{11}\} =$

μ_4. Here X_F^2 is not stable since x_{11} is a blocking contract, X_F^3 is not stable because it is not individually rational for w_5 as $C_{w_5}(\{x_3, x_6\}) = x_3$, and X_W^3 is not stable because it is not individually rational for f_2 as $C_{f_2}(\{x_4, x_6\}) = x_4$.

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